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Forum for CSIR-UGC JRF/NET, GATE, IIT-JAM/IISc, JEST, TIFR and GRE in PHYSICS & PHYSICAL SCIENCES

Waves, Oscillations & Optics

(IIT-JAM/JEST/TIFR/M.Sc Entrance)

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1. Simple Harmonic Oscillation

Periodic motion: A motion said to be periodic if, after some set amount of time it returns to its starting point and begins to repeat its motion.

Example: Earth rotation around sun, Movement of Clock Arm, Electron motion around nucleus, simple pendulum, and spring loaded mass etc.

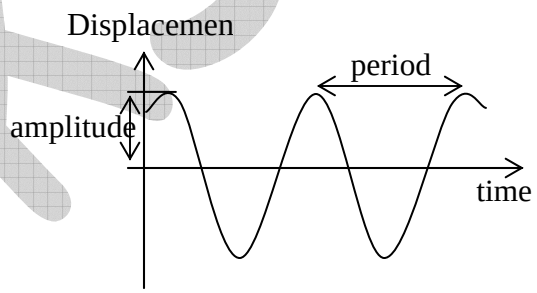
Simple Harmonic Motion: Simple harmonic motion is the motion of a mass when it is subject to the linear restoring force which is proportional but opposite to the displacement of the mass.

Example: Motion of spring loaded mass, simple pendulum, atomic vibration etc.

Important Terms

Cycle: The motion of the object from a starting position back to its starting position is called a cycle

Time Period: The time it takes to complete one cycle is called the time period (T).



Frequency: The frequency (f) is the number of cycles per unit time.

Amplitude: The maximum displacement of the mass from its equilibrium position is called the amplitude (A).

Phase: how much of a cycle the system completed (system that begins is in phase zero, system which completed half a cycle is in phase π).

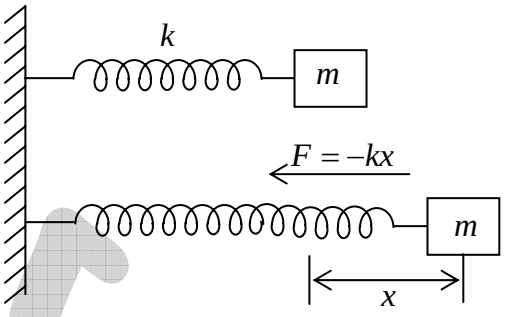
Equilibrium position: The position about which an object in harmonic motion oscillates; the center of vibration.

Hooke's law: law that states that the restoring force applied by a spring is proportional to the displacement of the spring and opposite in direction.

Restoring force: The force acting on an oscillating object which is proportional to the displacement and always points toward the equilibrium position.

1.1 Equations of Motion (Hooke's Law)

Consider a mass m is attached to the spring, and the other end of the spring is connected to a rigid support such as a wall. If the system is left at rest at the equilibrium position $(x=0)$ then there is no net force acting on the mass. However, if the mass is displaced from the equilibrium position, a restoring elastic force opposite to the displacement is exerted by the spring.



The equations of motion for a mass on a spring, using Hooke's law is

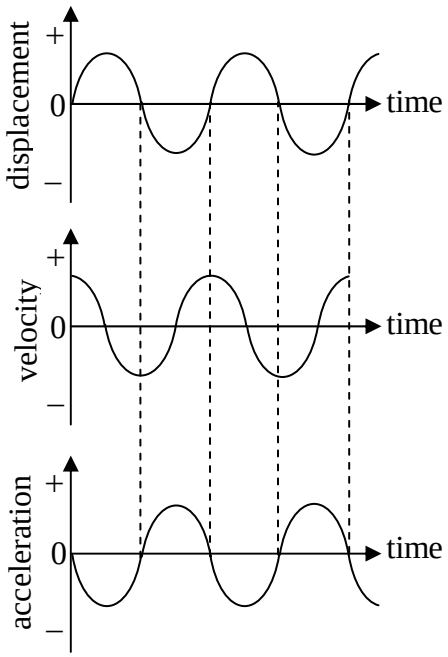
$$F = -kx$$

where F is the restoring elastic force exerted by the spring, k is the spring constant, and x is the displacement from the equilibrium position.

The equation of motion is a second-order linear ordinary differential equation obtained by means of Newton's second law.

$$F = m\frac{d^2x}{dt^2} = -kx \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}} \quad \text{is the Angular frequency of the oscillator.}$$



We see from this equation that the higher the spring constant k, the stiffer the spring, and the greater the angular frequency of oscillation. A smaller mass will also increase the angular frequency for a particular spring.

Solving the differential equation above, a solution which is a sinusoidal function is obtained.

$$x(t) = Ce^{i\omega t} + De^{i\omega t} = A\sin(\omega t + \phi)$$

The velocity and acceleration is

$$v(t)=\frac{dx}{dt}=A\omega\cos(\omega t+\phi)=\omega\sqrt{A^2-x^2} \text{ and } a(t)=\frac{d^2x}{dt^2}=-A\omega^2\sin(\omega t+\phi)=-\omega^2 x$$

Then since $\omega=2\pi f$, $f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$

and since $T=\frac{1}{f}$ where T is the time period, $T=2\pi\sqrt{\frac{m}{k}}$

Example: An object oscillates with simple harmonic motion along the x - axis. Its position varies with time according to the equation

$$x=(4.00\,m)\cos\left(\pi t+\frac{\pi}{4}\right)$$

where t is in seconds and the angles in the parentheses are in radians.

(a) Determine the amplitude, frequency, and period of the motion.

(b) Calculate the velocity and acceleration of the object at any time t.

(c) Using the results of part (b), determine the position, velocity, and acceleration of the object at $t=1.00\,s$.

(d) Determine the maximum speed and maximum acceleration of the object.

(e) Find the displacement of the object between $t=0$ and $t=1.00\ s$.

Solution:

(a) Since $x=A\cos(\omega t+\phi)$, we see that $A=4.00\,m$ and $\omega=\pi\,\text{rad}/s$.

Therefore, $f=\omega/2\pi=\pi/2\pi=0.500\,Hz$ and $T=1/f=2.00\,s$.

(b) Differentiating x to find v to find a, we obtain

$$v=\frac{dx}{dt}=-(4.00\pi\,m/s)\sin\left(\pi t+\frac{\pi}{4}\right) \text{ and } a=\frac{dv}{dt}=-\left(4.00\pi^2\,m/s^2\right)\cos\left(\pi t+\frac{\pi}{4}\right)$$

(c) At $t=1.00\,s$,

$$x=(4.00\,m)\cos\left(\pi+\frac{\pi}{4}\right)=(4.00\,m)\cos\left(\frac{5\pi}{4}\right)=(4.00\,m)(-0.707)=-2.83\,m$$

$$v = -(4.00\pi\, m/s)\sin\left(\frac{5\pi}{4}\right) = -(4.00\pi\, m/s)(-0.707) = 8.89\, m/s$$

$$a = -(4.00\pi^2\, m/s^2)(-0.707) = 27.9\, m/s^2$$

(d) Since maximum values of the sine and cosine functions are unity. Therefore, v varies between $\pm 4.00\pi\, m/s$ and a varies between $\pm 4.00\pi^2\, m/s^2$.

Thus, $v_{max} = 4.00\pi\, m/s = 12.6\, m/s$ and $a_{max} = 4.00\pi^2\, m/s^2 = 39.5\, m/s^2$

We obtain the same results using the relations $v_{max} = \omega A$ and $a_{max} = \omega^2 A$ where $A = 4.00\, m$ and $\omega = \pi\, \text{rad}/s$

(e) The position at $t = 0$ is

$$x_i = (4.00\, m)\cos\left(0 + \frac{\pi}{4}\right) = (4.00\, m)(0.707) = 2.83\, m$$

In part (c), we found that the position at $t = 1.00\, s$ is $-2.83\, m$; therefore, the displacement between $t = 0$ and $t = 1.00\, s$ is

$$\Delta x = x_f - x_i = -2.83\, m - 2.83\, m = -5.66\ m$$

Because the object's velocity changes sign during the first second, the magnitude of Δx is not the same as the distance traveled in the first second. (By the time the first second is over, the object has been through the point $x = -2.83\, m$ once, traveled to $x = -4.00\ m$ and come back to $x = -2.83\, m$)

1.2 Energy of Simple Harmonic Motion

As an object vibrates in harmonic motion, energy is transferred between potential energy and kinetic energy. If we stretch a spring from its equilibrium (unstretched) position to a certain displacement the work done is equal to the stored potential energy in the spring. If we release the mass and allow it to begin moving back toward the equilibrium position, the potential energy begins changing into kinetic energy. As the mass passes through the equilibrium position, all of the potential energy has been converted into kinetic energy, and the speed of the mass is maximum. The kinetic energy in turn begins changing into potential energy, until all of the kinetic energy is converted into potential energy at

maximum compression. The compressed spring then accelerates the mass back through the equilibrium to the original starting position, and the entire process repeats itself. The total energy of the system remains constant, that is,

Total Energy = Potential Energy + Kinetic Energy = a constant

The kinetic energy K of the system at time t is

$$K=\frac{1}{2}mv^2=\frac{1}{2}m\omega^2A^2\cos^2(\omega t+\phi)=\frac{1}{2}kA^2\cos^2(\omega t+\phi)=\frac{1}{2}k\left(A^2-x^2\right)$$

The potential energy is $U=\frac{1}{2}kx^2=\frac{1}{2}kA^2\sin^2(\omega t+\phi)$

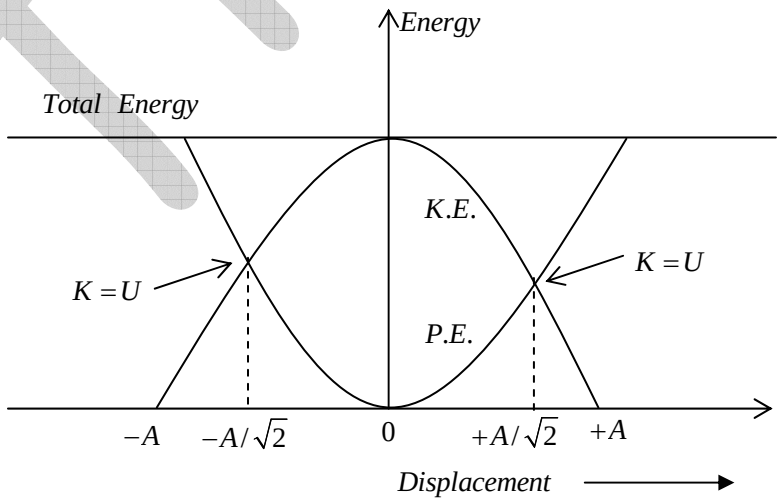
The total mechanical energy of the system therefore has the constant value

$$E=K+U=\frac{1}{2}kA^2$$

The total energy is constant and has the value is $\frac{1}{2}kA^2$. This is also the maximum value of the potential or the kinetic energy. Thus at the time when P.E. is zero the K.E. is maximum or vice versa.

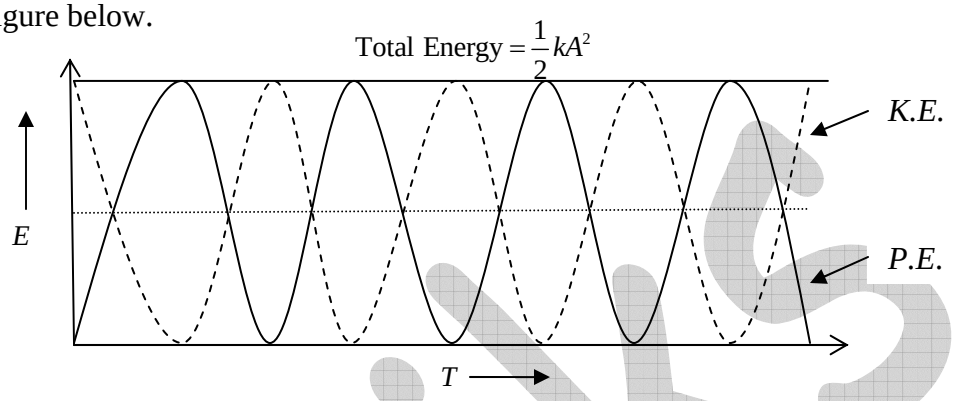
Energy vs Position Graph

The distribution of energy versus displacement for S.H.M. is shown in the figure. The P.E. is a parabola with vertex at $x=0$, so the energy is stored in the oscillator both when x is positive and negative. The K.E. curve is an inverted parabola.



Energy vs Time Graph

The distribution of energy versus time for S.H.M. is shown in the figure. The P.E. $U=\frac{1}{2}kA^2\sin^2(\omega t+\phi)$ and K.E. is $K=\frac{1}{2}kA^2\cos^2(\omega t+\phi)$ variation with time is shown in figure below.



Frequency of energy change in S.H.M.

The energy of the particle in S.H.M. changes periodically from P.E. to K.E. Both U and K depends upon the square of the displacement x, i.e. each acquires same value twice in each oscillation. Thus the frequency of energy change is twice the frequency of oscillation.

Time average of kinetic and potential energy

Time average of kinetic energy $K(t)_{av}=\frac{\int_0^T K(t)\,dt}{T}=\frac{\int_0^T\frac{1}{2}kA^2\cos^2(\omega t+\phi)\,dt}{T}=\frac{1}{4}kA^2$

Time average of potential energy $U(t)_{av}=\frac{\int_0^T U(t)\,dt}{T}=\frac{\int_0^T\frac{1}{2}kA^2\sin^2(\omega t+\phi)\,dt}{T}=\frac{1}{4}kA^2$

Thus the time average of kinetic energy is equal to the time average of the potential energy which is half the total energy.

Position average of kinetic and potential energy

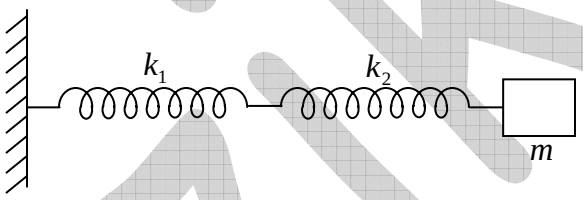
Position average of kinetic energy $K(x)_{av} = \dfrac{\int_0^A K(x)\,dt}{A} = \dfrac{\int_0^A \frac{1}{2}k\left(A^2 - x^2\right)}{A} = \dfrac{1}{3}kA^2$

Position average of potential energy $U(x)_{av} = \dfrac{\int_0^T U(x)\,dt}{A} = \dfrac{\int_0^A \frac{1}{2}kx^2 dx}{A} = \dfrac{1}{6}kA^2$

Thus the position average of kinetic energy is equal to the twice of the position average of the potential energy.

1.3 Mass Loaded Spring System

Example: Find the time period for the arrangement for the following mass loaded spring having spring constant k_1 and k_2.



Solution: Let x be the displacement of m from its equilibrium position at an instant, and x_1, x_2 is extension in the length of springs k_1 and k_2 respectively.

Thus $x = x_1 + x_2$

The restoring force in k_1 and k_2 will be the same. By Hooke's law we have

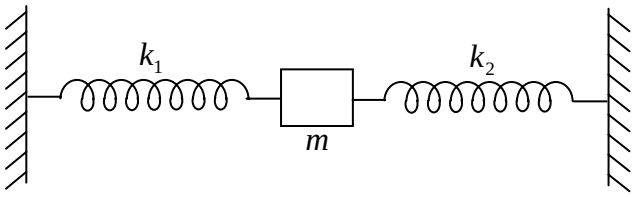
$$F = -k_1 x_1 = -k_2 x_2$$

$$x = -\frac{F}{k_1} - \frac{F}{k_2} \Rightarrow F = -\frac{k_1 k_2}{k_1 + k_2} x = -kx$$

Thus the force constant of the system is $k = \dfrac{k_1 k_2}{k_1 + k_2}$.

Hence the time period is $T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{(k_1 + k_2)m}{k_1 k_2}}$

Example: Find the time period for the arrangement for the following mass loaded spring having spring constant k_1 and k_2.



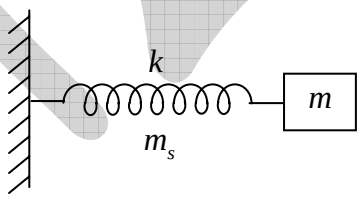
Solution: During oscillation one spring stretched and other compressed and vice versa. Let x be the displacement of mass m from its equilibrium distance, and this is also the compression and expansion in the springs k_1 and k_2. The restoring force developed in the two springs is $F_1 = -k_1 x$ and $F_2 = -k_2 x$. F_1 and F_2 act on the mass in the same direction

$$F = F_1 + F_2 = -k_1 x - k_2 x = -\left(k_1 + k_2\right) x = -kx$$

Thus the force constant of the system is $\left(k_1 + k_2\right)$, and the time period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

Example: Determine the time period of the system of mass m attached with spring of mass m_s $\left(m_s << m\right)$ and spring constant k.



Solution: If $m_s << m$, then the spring will stretch uniformly along its length. Let l be the length of the spring, its mass per unit length will be $\frac{m_s}{l}$. Let us consider an element of length ds at a distance s from the fixed end of the string. The mass of this element is $\left(\frac{m_s}{l}\right) ds$. Let x be the instantaneous displacement of m.

The displacement of a element of spring $=\left(\frac{s}{l}\right)x$

The instantaneous velocity of element $=\left(\frac{s}{l}\right)\frac{dx}{dt}$

The instantaneous kinetic energy of element $=\frac{1}{2}\left(\frac{m_s}{l}ds\right)\left(\frac{sdx}{ldt}\right)^2=\frac{m_s}{2l^3}\left(\frac{dx}{dt}\right)^2 s^2 ds$

The total kinetic energy of the uniform spring $\frac{m_s}{2l^3}\left(\frac{dx}{dt}\right)^2\int_0^l s^2 ds=\frac{1}{2}\left(\frac{m_s}{3}\right)\left(\frac{dx}{dt}\right)^2$

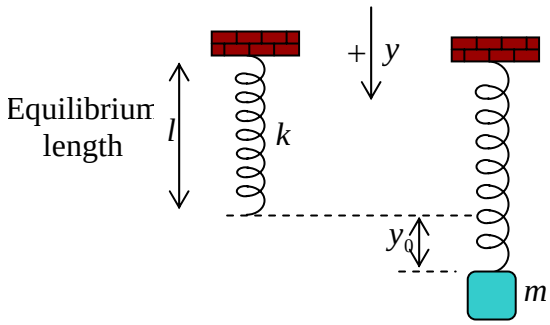
The total kinetic energy of the whole system is $K=\frac{1}{2}\left(m+\frac{m_s}{3}\right)\left(\frac{dx}{dt}\right)^2$

This shows the effective mass of the system is $m+\frac{m_s}{3}$.

Hence the time period is $T=2\pi\sqrt{\frac{m+\frac{m_s}{3}}{k}}$

Example: Find the time period for the arrangement for the following mass loaded vertical spring having spring constant k .

Solution: Consider l be the length of the spring hanging vertically. When mass m attached to the lower end, it's length extended by y_0 . By Hooke's law the restoring force is $F=-ky_0$.



The other force acting on the mass is its weight $+mg$ (downward). Thus the total force on m is $-ky_0+mg$.

Since mass has no acceleration, the total force on mass m is zero,

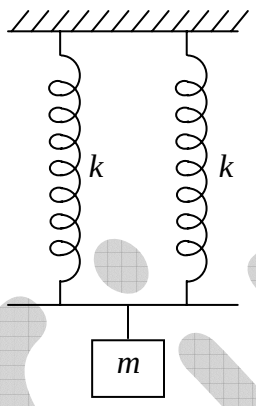
$$-ky_0+mg=0\Rightarrow y_0=mg/k$$

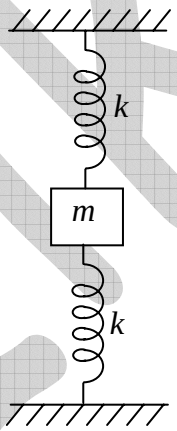
When the body is pulled through a small distance y from the equilibrium position and released, it start oscillating with SHM, since the restoring force is $F = -ky$, imparting an acceleration of $\frac{d^2y}{dt^2}$ is given by $\frac{d^2y}{dt^2} = -\frac{k}{m}y$

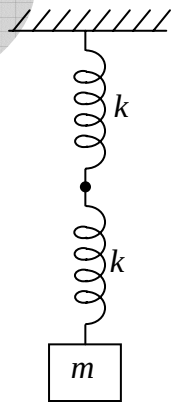
Which is a equation of SHM whose time period is $T = 2\pi\sqrt{\frac{m}{k}}$

It can also be written as $T = 2\pi\sqrt{\frac{y_0}{g}}$

Example: Find the time period of the following sets of mass loaded springs hanged vertically.

(a) 

(b) 

(c) 

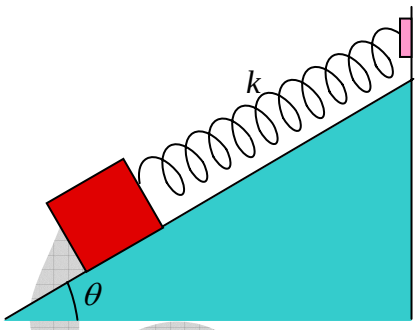
Solution:

(a) Since both springs are in parallel combination. The resultant spring constant of the system is $k' = k + k = 2k$. Thus the time period is $T = 2\pi\sqrt{\frac{m}{2k}}$

(b) Since both springs are in series. The resultant spring constant of the system is $k' = \frac{k.k}{(k+k)} = \frac{k}{2}$. Thus the time period is $T = 2\pi\sqrt{\frac{2m}{k}}$

(c) In this case also springs are in series. Thus time period of the system is $T = 2\pi\sqrt{\frac{2m}{k}}$

Example: A block weighing$14.0\ N$, which can slide without friction on an incline at angle $\theta = 40.0^0$, is connected to the top of the incline by a massless spring of unstretched length $0.450\ m$ and spring constant$120\frac{N}{m}$.



(a) How far from the top of the incline is the block's equilibrium point?

(b) If the block is pulled slightly down the incline and released, what is the period of the resulting oscillations?

Solution:

(a) The equilibrium position when the block is gently lowered until forces balance. If the amount spring is stretched is x, then we examine force-components along the incline surface and find

$$kx = mg\sin\theta \Rightarrow x = \frac{14.0\sin 40.0^0}{120} = 0.0750\ m$$

At equilibrium, the distance from the top of the incline is therefore

$$(0.450 + 0.075)m = 0.525\ m.$$

(b) Just as with a vertical spring, the effect of gravity (or one of its components) is simply to shift the equilibrium position; it does not change the characteristics (such as the period) of simple harmonic motion.

$$T = 2\pi\sqrt{\frac{14.0}{\frac{9.80}{120}}} = 0.686\ s$$

Example: A spring of constant $k = 100\dfrac{N}{m}$ hangs at its natural length from a fixed stand.

A mass of $3\ kg$ is hung on the end of the spring, and slowly let down until the spring and mass hang at their new equilibrium position.

(a) Find the value of the quantity x in the figure above.

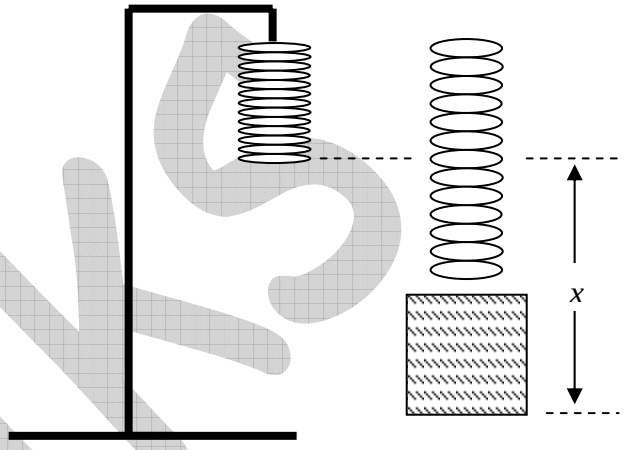
The spring is now pulled down an additional distance x and released from rest.

(b) What is the potential energy in the spring at this distance?

(c) What is the speed of the mass as it passes the equilibrium position?

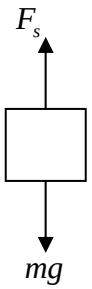
(d) How high above the point of release will the mass rise?

(e) What is the period of oscillation for the mass?



Solution:

(a) As it hangs in equilibrium, the upward spring force must be equal and opposite to the downward weight of the block.



$$F_s = mg \text{ and } kx = mg \Rightarrow x = \frac{mg}{k} = \frac{(3\ \ kg)(10\ \ m/s^2)}{100\ \ N/m} = 0.3\ \ m$$

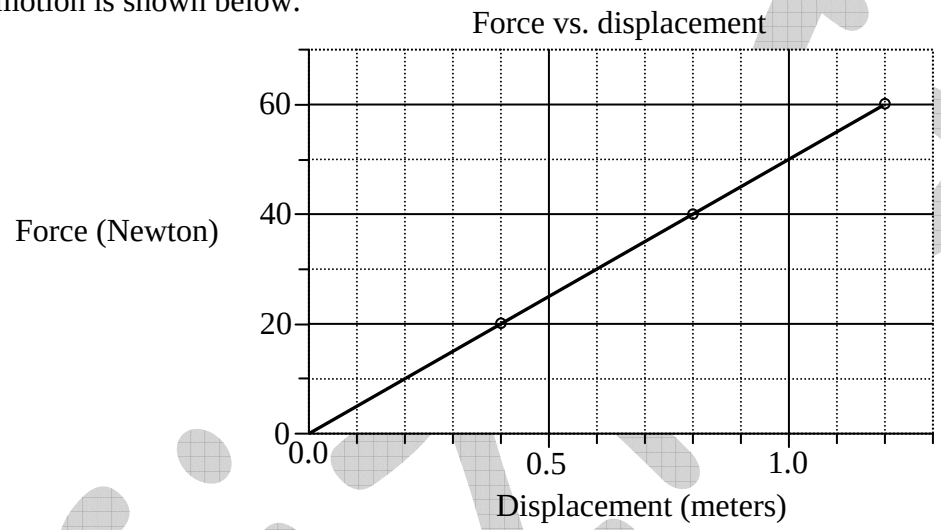
(b) The potential energy in the spring is related to the displacement from equilibrium position by the equation $U = \dfrac{1}{2}kx^2 = \dfrac{1}{2}(100N/m)(0.3m)^2 = 4.5J$

(c) Since energy is conserved during the oscillation of the mass, the kinetic energy of the mass as it passes through the equilibrium position is equal to the potential energy at the amplitude. Thus, $K = U = \dfrac{1}{2}mv^2 \Rightarrow v = \sqrt{\dfrac{2U}{m}} = \sqrt{\dfrac{2(4.5\ \ J)}{3\ \ kg}} = 1.7m/s$

(d) Since the amplitude of the oscillation is $0.3\ m$, it will rise to $0.3\ m$ above the equilibrium position.

(e) $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{3kg}{100\,N/m}} = 1.1s$

Example: A mass of $0.5\ kg$ oscillates on the end of a spring on a horizontal surface with negligible friction according to the equation $x = A\cos(\omega t)$. The graph of F *vs* x for this motion is shown below.



The last data point corresponds to the maximum displacement of the mass. Determine the

(a) Angular frequency ω of the oscillation,

(b) Frequency f of oscillation,

(c) Amplitude of oscillation,

(d) Displacement from equilibrium position $(x = 0)$ at a time of $2s$.

Solution: (a) When we looked at this graph $k = 50\frac{N}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{50\frac{N}{m}}{0.5\ kg}} = 10\frac{rad}{s}$

(b) $f = \frac{\omega}{2\pi} = \frac{10\ \ rad/s}{2\pi} = 1.6\ \ Hz$

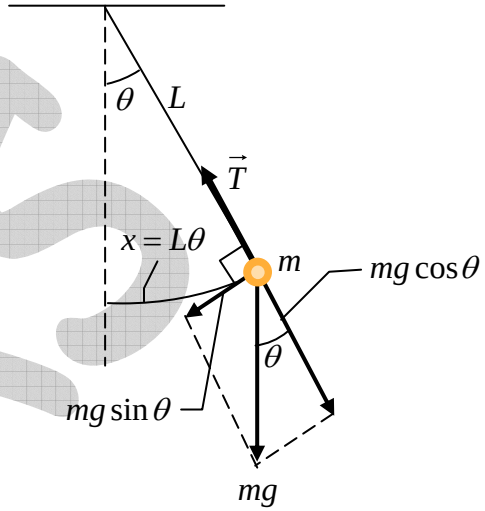
(c) The amplitude corresponds to the last displacement on the graph, $A = 1.2\ m$.

(d) $x = A\cos(\omega t) = (1.2\ \ m)\cos\left[(10\ \ rad/s)(2\ \ s)\right] = 0.5m$

1.4 Simple Pendulum

A simple pendulum is an idealized system containing of a massless inextensible string, fixed rigidly at one end, having a point mass at the other end.

A mass m suspended by a light string of length L from a fixed point. When the mass is displaced slightly from its equilibrium position and released, it oscillates in a vertical plane under gravity. Let θ be the angular displacement of the pendulum. The force acting on m is mg. This force can be resolved into two components, a radial component $mg\cos\theta$ along the string and tangential component $mg\sin\theta$ at right angle to it. The tangential component is the restoring force.



Thus $F = -mg\sin\theta$

Using Newton's law, we get $m\dfrac{d^2x}{dt^2} = -mg\sin\theta$

For small angles of oscillations $\sin\theta = \dfrac{x}{L}$, is governed by $m\dfrac{d^2x}{dt^2} = -mg\dfrac{x}{l}$

The frequency (f) and the period (T) of the pendulum in this case are given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \quad \text{and} \quad T = 2\pi\sqrt{\frac{l}{g}}$$

Therefore, the simple pendulum has the following properties

(i) The period of a pendulum is independent of its own mass and the amplitude of the oscillation.

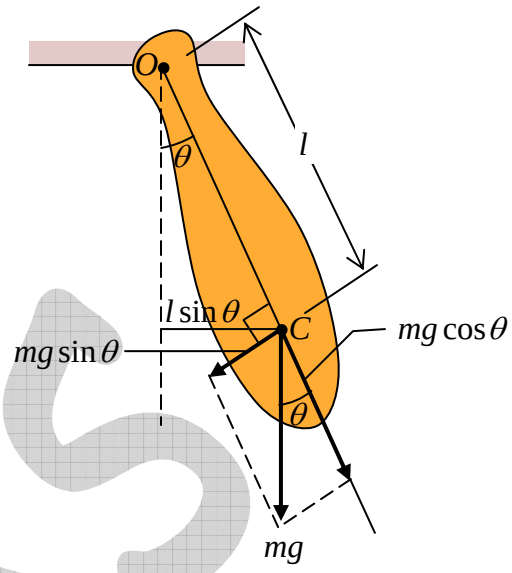
(ii) The period of the pendulum is depends on its length.

Large Amplitude: When the amplitude of motion of the simple pendulum is large, the approximation $\sin q = q$ does not hold and the pendulum deviates from the simple harmonic behavior. Then the time period $T = 2\pi\sqrt{\left(\dfrac{l}{g}\right)\left(1+\dfrac{1}{2}\sin^2\dfrac{\theta}{2}+\dfrac{1}{2^2}\dfrac{3^2}{4^2}\sin^4\dfrac{\theta}{2}+\ldots\right)}$

Thus with increasing amplitude the time period increases.

1.5 Compound Pendulum

A compound pendulum is a rigid body, of any shape, capable of oscillating about a horizontal axis passing through it.



Let m be the mass of the body and l be the distance of C from O. When body displaced by an angle θ with the vertical, at this instant the **moment** of its weight mg about the axis of oscillation through O is $mg(l\sin\theta)$.

This is the restoring torque which brings the body to its equilibrium position.

Thus $\tau = -mgl\sin\theta$

For small angle, we have; $\tau = -mgl\theta$

If I is the moment of inertia of the pendulum about the axis of suspension O, the torque is also equal to

$$I\frac{d^2\theta}{dt^2} = -mgl\theta \quad \Rightarrow \frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\theta$$

Thus the time period is $T = 2\pi\sqrt{\dfrac{I}{mgl}}$

It is sometime more useful to write the time period in term of radius of gyration of pendulum. If we write the moment of inertia about a parallel axis through C, then

$$I = mk^2 + ml^2 = m\left(k^2 + l^2\right)$$

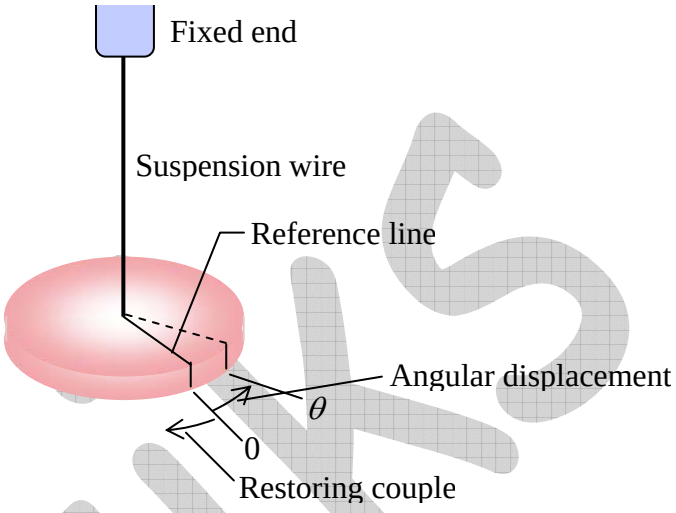
Thus the time period is

$$T = 2\pi\sqrt{\frac{\left(\frac{k^2}{l} + l\right)}{g}} = 2\pi\sqrt{\frac{L}{g}}$$

Here L is called the "length of the equivalent simple pendulum."

1.6 The Torsional Pendulum

A torsional pendulum consist of a heavy body such as disc or cylinder be hung by a long and thin vertical wire from a rigid support. If the disc is turned in the horizontal plane and then released, due to wire elasticity, it executes a torsional oscillation about the wire as a axis.



If the disc is turned by an angle θ, the wire is also twisted by same angle. A restoring torque $(-\tau\theta)$ produced which brings the pendulum back to its original position. If I is the moment of inertia of the disc about the wire, then the torque acting on the disc must be equal to the product of the moment of inertia I and the angular acceleration.

Thus $$\frac{d^2\theta}{dt^2} = -\tau\theta \quad \Rightarrow \frac{d^2\theta}{dt^2} = -\frac{\tau}{I}\theta$$

The time period of the oscillation is $T = 2\pi\sqrt{\frac{I}{\tau}}$

MCQ(Multiple Choice Questions)

Q1. A particle moving with simple harmonic motion passes through points P and Q with the same velocity. Having taken $2s$ in passing from P to Q, it returns to Q after another $2s$. The period (in seconds) is:

(a) 2 (b) 4 (c) 6 (d) 8

Q2. The angular frequency of motion whose equation is $4\frac{d^2y}{dt^2}+9y=0$ is:

($y=$ Displacement and $t=$ time)

(a) $\frac{9}{4}$ (b) $\frac{4}{9}$ (c) $\frac{3}{2}$ (d) $\frac{2}{3}$

Q3. The displacement of a particle executing simple harmonic motion is given by $y=4\sin\left(2t+\phi\right)$. The period of oscillation is:

(a) $\frac{2}{\pi}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π

Q4. The equation of wave is given by $y=10\sin\left(\frac{2\pi t}{30}+\alpha\right)$

If the displacement is $5\ cm$ at $t=0$, then the total phase at $t=7.5\ s$ will be:

(a) $\frac{2\pi}{3}$ rad (b) $\frac{2\pi}{5}$ rad (c) $\frac{2\pi}{4}$ rad (d) $\frac{\pi}{2}$ rad

Q5. When a harmonic wave is propagating through a medium, the displacement y of a particle of the medium is $y=10\sin\frac{2\pi}{5}\left(1800t-x\right)$. Then the time period will be:

(a) $\frac{1}{360}s$ (b) $\frac{1}{36}s$ (c) 36 s (d) 360s

Q6. A particle executes simple harmonic motion between $x=-A$ and $x=+A$. The time taken for it to go from 0 to $\frac{A}{2}$ is T_1 and to go from $\frac{A}{2}$ to A is T_2. Then

(a) $T_1<T_2$ (b) $T_1>T_2$ (c) $T_1=T_2$ (d) $T_1=2T_2$

Q7. A thick spherical shell is filled with water. It is hung by a long thread and the water slowly flows out of a hole at the bottom. The period of oscillation of spherical shall:

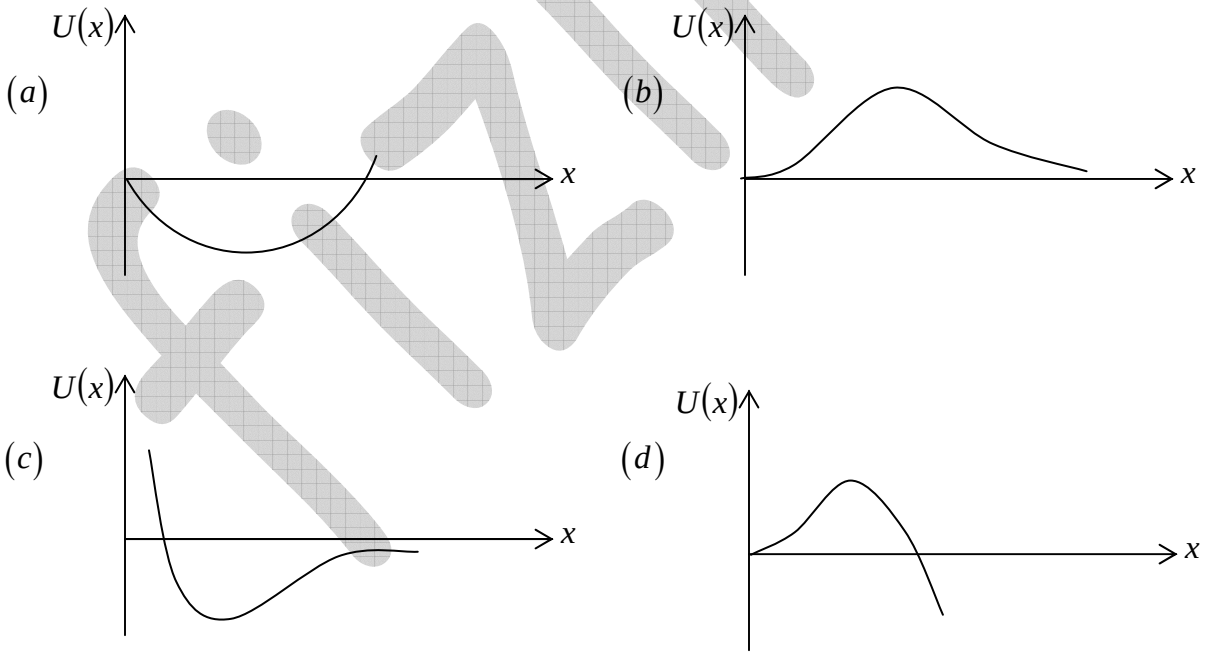
(a) continuously decrease (b) first decrease and than increase

(c) continuously increase (d) first increase and then decrease

Q8. A particle executes simple harmonic motion with a frequency f. The frequency with which its kinetic energy oscillates is:

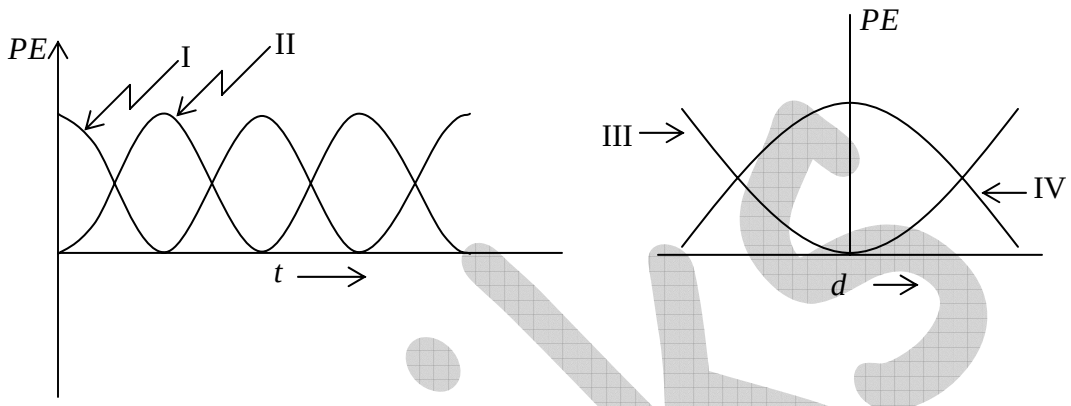
(a) $\frac{f}{2}$ (b) f (c) $2f$ (d) $4f$

Q9. A particle, which is constrained to move along the x -axis, is subjected to a force in the same direction which varies with the distance x of the particle from the origin as $F(x) = -kx + ax^3$. Here k and a are positive constants. For $x \geq 0$, the functional form of the potential energy $U(x)$ of the particle is



Q10. For a particle executing simple harmonic motion the displacement x is given by $x = A\cos\omega t$. Identify the graph which represents the variation of potential energy (PE) as a function of time and displacement

(a) I, III (b) II, IV (c) II, III (d) I, IV

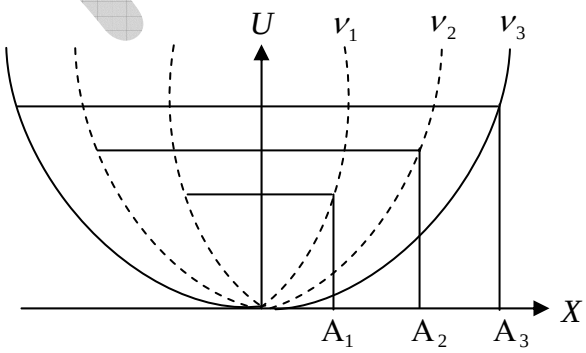


Q11. Three simple harmonic oscillators, with amplitudes $A_1 > A_2 > A_3$, pass through their respective equilibrium positions with the same velocity. If these oscillators have the same mass, then their characteristic frequencies ω_1, ω_2 and ω_3 will be related to each other as:

(a) $\omega_1 A_1 = \omega_2 A_2 = \omega_3 A_3$ (b) $\omega_1^2 A_1 = \omega_2^2 A_2 = \omega_3^2 A_3$

(c) $\omega_1 A_2^2 = \omega_2 A_2^2 = \omega_3 A_3^2$ (d) $(\omega_1^2 / A_1) = (\omega_2 / A_2) = (\omega_3 / A_3)$

Q12. The potential energy, U of three simple harmonic oscillators varies with displacement x as shown in the figure. These oscillators have the same mass but different amplitudes A_1, A_2 and A_3. Their characteristic frequencies ν_1, ν_2 and ν_3 are related to each other as:



(a) $\frac{\nu_1}{A_1}=\frac{\nu_2}{A_2}=\frac{\nu_3}{A_3}$ (b) $\nu_1 A_1=\nu_2 A_2=\nu_3 A_3$

(c) $\nu_1>\nu_2>\nu_3$ (d) $\nu_1<\nu_2<\nu_3$

Q13. A mass of $1\,kg$ suspended from a spring whose force constant is $400\ Nm^{-1}$, executes simple harmonic oscillation. When the total energy of the oscillator is $2J$, the maximum acceleration experienced by the mass will be:

(a) $2ms^{-2}$ (b) $4ms^{-2}$ (c) $40ms^{-2}$ (d) $400\ ms^{-2}$

Q14. Two blocks each of mass m are connected to the ends of a massless rod of length $2l$ which is rotated in a horizontal plane about its vertical axis of symmetry. The maximum frequency with which the system can be rotated without exceeding the breaking tension T for the rod is:

(a) $\frac{1}{2\pi}\left[\frac{T}{ml}\right]^{1/2}$ (b) $\frac{1}{2\pi}\left[\frac{ml}{T}\right]^{1/2}$

(c) $\frac{1}{2\pi}\left[\frac{2T}{ml}\right]^{1/2}$ (d) $\frac{1}{2\pi}\left[\frac{T}{2ml}\right]^{1/2}$

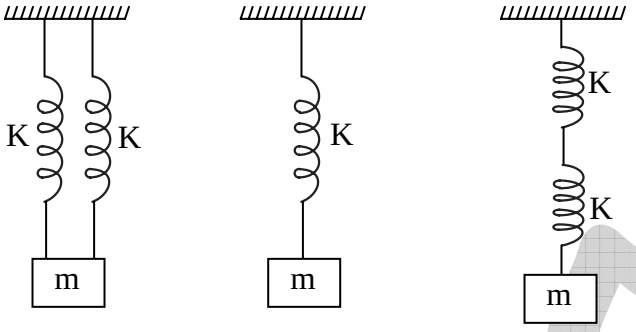
Q15. The period of oscillation of a mass, M suspended from a spring of negligible mass is T. If along with it another mass M is also suspended the period of oscillation will now be:

(a) T (b) $\frac{T}{\sqrt{2}}$ (c) $2T$ (d) $\sqrt{2}T$

Q16. A spring of force-constant k is cut into two pieces such that one piece is double the length of the other. Then the long piece will have a force constant of

(a) $\left(\frac{2}{3}\right)k$ (b) $\left(\frac{3}{2}\right)k$ (c) $3k$ (d) $6k$

Q17. In the case of suspensions shown in the given diagrams, the ratio of the time period of the three simple harmonic oscillators will be:



(a) $\sqrt{2}:1:\frac{1}{\sqrt{2}}$ (b) $2:1:\frac{1}{2}$

(c) $\frac{1}{2}:1:2$ (d) $\frac{1}{\sqrt{2}}:1:\sqrt{2}$

Q18. An ideal spring with spring-constant k is hung from the ceiling and a block of mass M is attached to its lower end. The mass is released with the spring initially unstretched. Then the maximum extension in the spring is

(a) $\frac{4Mg}{k}$ (b) $\frac{2Mg}{k}$ (c) $\frac{Mg}{k}$ (d) $\frac{Mg}{2k}$

Q19. A simple pendulum attached to the ceiling of a stationary lift has a time period T. When the lift moves upward with distance covered as $y=\left(1m/s^2\right)t^2$, the time period of the pendulum will be equal to (Take $g=10.0\ m/s^2$)

(a) $\sqrt{\frac{4}{5T}}$ (b) $\sqrt{\frac{5}{6T}}$ (c) $\sqrt{\frac{5}{4T}}$ (d) $\sqrt{\frac{6}{5T}}$

Q20. A simple pendulum of time period T is taken to a place where the value of g be 0.8% more than the value of g on the surface of the earth, the percentage change in the time period of a simple pendulum is

(a) 0.1% (b) 0.2% (c) 0.3% (d) 0.4%

Q21. A simple pendulum of length 'l' is suspended from the root of a train which moves in a horizontal direction with an acceleration 'a'. The one time period T is given by

(a) $2\pi\sqrt{\frac{l}{g}}$ (b) $2\pi\sqrt{\frac{l}{\left(a^2+g^2\right)^{\frac{1}{2}}}}$

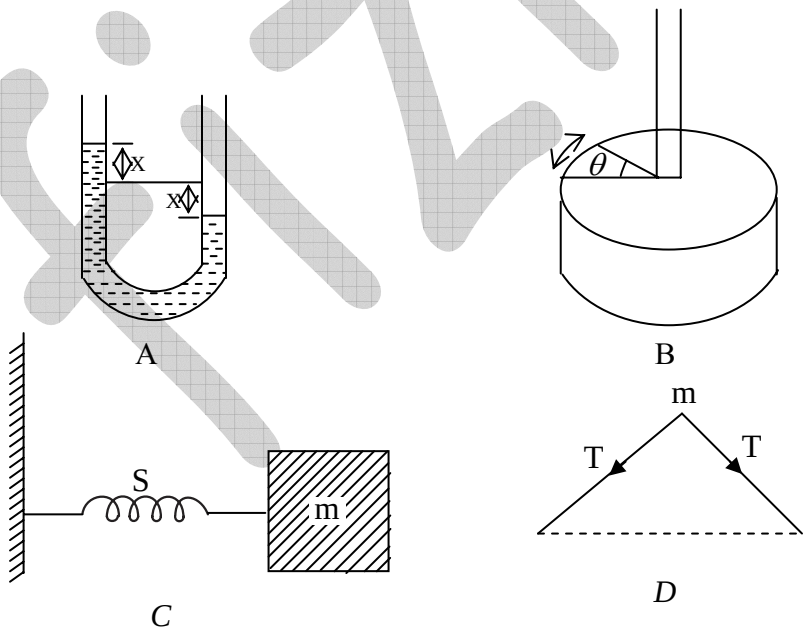
(c) $2\pi\sqrt{\frac{l}{g}}$ (d) $2\pi\sqrt{\frac{l}{g-a}}$

Q22. The period of a simple pendulum measured inside a stationary lift is found to be T. If the lift start accelerating upward with acceleration of $\frac{g}{3}$, then the time period of the pendulum is

(a) $\frac{2T}{\sqrt{3}}$ (b) $\frac{T}{3}$ (c) $\frac{\sqrt{3}}{2}T$ (d) $\sqrt{3}T$

MSQ (Multiple Select Questions)

Q23. Consider the mechanical vibrating systems shows in figures A, B, C and D:



The vibrations are simple harmonic in:

(a) A (b) B (c) C (d) D

Q24. The total energy of a particle executing simple harmonic motion depends on its:

(a) Amplitude (b) Period

(c) Displacement (d) mass

Q25. For a particle executing S.H.M, which statements are correct?

(a) The velocity of the particle is maximum at the mean position

(b) The sum of the kinetic and potential energies of the particle at any instant remain constant

(c) The acceleration of particle is maximum at the extreme position

(d) the restoring force is always direct towards the mean position

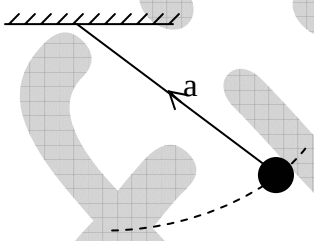
Q26. Which of the following functions represents S.H.M?

(a) $\sin^2 \omega t$ (b) $\cos 2\omega t$

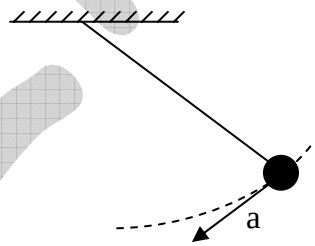
(c) $\sin \omega t - \cos \omega t$ (d) $\sin \omega t + 2\cos \omega t$

Q27. A simple pendulum is oscillating without damping. When the displacement of bob is less than maximum, its acceleration vector $\vec{a}$ is correctly shown in:

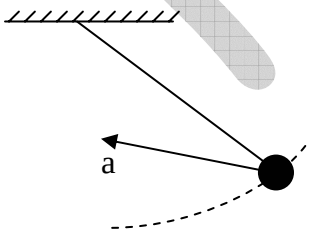
(a)



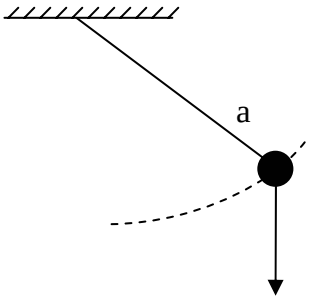
(b)



(c)



(d)



.

Q28. A spring pendulum and a simple pendulum have time periods $2s$ and $3s$ respectively on the surface of the earth. These pendulum are then taken to the surface of the moon, then

(a) Time period of the simple pendulum is same on the surface of the moon

(b) Time period of the spring pendulum is same on the surface of the moon

(c) Time period of the simple pendulum is less than $3s$ on the surface of the moon

(d) Time period of the simple pendulum is more than $3s$ on the surface of the moon

Q29. If the amplitude of a particle executing S.H.M. is doubled, which of the quantities will be doubled?

(a) total energy (b) time period

(c) maximum velocity (d) maximum acceleration

Q30. A simple harmonic motion is represented by $y = a\sin\omega t$. Which of the following statements are true?

(a) The phase difference between displacement and velocity is $\frac{\pi}{2}$

(b) The phase difference between displacement and acceleration is π

(c) The magnitude of maximum acceleration is $\frac{a\omega^2}{2}$

(d) The maximum velocity of the particle is at the extreme position

NAT (Numerical Answer Type)

Q31. A simple pendulum has a time period T_1 when on the earth's surface, and T_2 when taken to a height R above the earth's surface, where R is the radius of the earth. The value of $\frac{T_2}{T_1}$ is .....

Q32. If the earth were a homogeneous sphere and a straight hole were bored in it through its centre. A ball dropped into it executes SHM. The time period of the oscillation is ……..sec.

Q33. A simple pendulum $2.0\ m$ in length oscillates at location where $g = 9.8 m/\sec^2$. The number of oscillations will it make in 5 minutes is……

Q34. If the length of the pendulum is increased by 4%, then the percentage change in the time of a simple pendulum is ……. %.

Q35. Two masses $m_1 = 1.0\ kg$ and $m_2 = 0.5\ kg$ are suspended together by a massless spring to spring constant k as shown below. When masses are in equilibrium m_1 is removed without disturbing the system. Assuming $k = 12.5\ N/m$, the amplitude of oscillation of m_2 is ……cm

Q36. The mass m attached to a spring period oscillates with a period of $2s$. If the mass is increased by $2kg$, the period increases by $1s$. The initial mass m is …..….. kg (assume Hook's law is obeyed)

Q37. A particle executes simple harmonic oscillation with an amplitude 'a'. The period of oscillation is 12 sec. The minimum time taken by the particle to travel half of the amplitude from the equilibrium position is ……..…sec

Q38. A spring (spring constant $= 50\ N/m$) is cut into 4 equal parts and the two parts are connected in parallel. The effective spring constant is……. N/m

Q39. The pendulum bob has a speed of $3\ m/s$ at its lowest position. The pendulum is $0.5\ m$ long. The speed of the bob, when the length makes an angle of 60° to the vertical will be ……..……. $m/\sec\left(g = 10 ms^{-2}\right)$

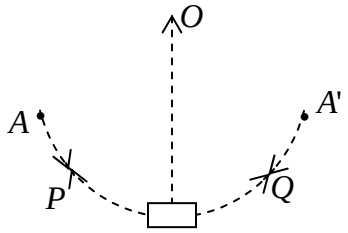
Q40. A particle of mass $5\,g$ is excuting S.H.M. with an amplitude $0.3\,m$ and time period $\frac{\pi}{5}$ sec. The maximum value of force acting on the particle is …….. N.

Q41. A simple pendulum is executing simple harmonic motion with a time period T. If the length of the pendulum is increases by 21%, the increase in the time period of the pendulum of increased length is ……%

fiziks

Solution-MCQ

Ans. 1: (d) Solution: Total time period is T, then $\frac{1}{2}T = t_{AP} + t_{PQ} + t_{QA'}$



$v_p = v_Q \Rightarrow t_{AP} = t_{QA'} \Rightarrow \frac{1}{2}T = t_{PQ} + 2t_{QA'}$

$\frac{1}{2}T = 2+2 \Rightarrow \frac{1}{2}T = 4 \Rightarrow T = 2\times 4 = 8s \because t_{QA'} = 1s$ and $t_{PQ} = 2s$

Ans. 2: (c)

Solution: The equation of SHM of angular velocity is given as $\ddot{y} + \omega^2 y = 0$ ---------- (i)

we have here $4\frac{d^2y}{dt^2} + 9y = 0 \Rightarrow 4\ddot{y} + 9y = 0 \Rightarrow \ddot{y} + \left(\frac{9}{4}\right)y = 0$ --------- (ii)

By equation (i) and (ii), we get $\omega^2 = \frac{9}{4} \Rightarrow \omega = \frac{3}{2}$ and $\frac{2\pi}{T} = \frac{3}{2} \Rightarrow T = \frac{4\pi}{3}$

Ans. 3: (c) Solution: $\because y = 4\sin(2t+\phi) \Rightarrow \omega = 2 = \frac{2\pi}{T} \Rightarrow T = \pi$

Ans. 4: (a)

Solution: Equation of waves is given as $y = 10\sin\left(\frac{2\pi t}{30} + \alpha\right)$

at $t = 0, y = 5 \Rightarrow 5 = 10\sin\alpha \Rightarrow \sin\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{6}$

Now, total phase at $7.5\ s$

$\phi = \frac{2\pi t}{30} + \alpha = \frac{2\pi}{30}\times 7.5 + \frac{\pi}{6} \Rightarrow \phi = \frac{2\pi}{4} + \frac{\pi}{6} \Rightarrow \phi = \frac{6\pi + 2\pi}{12} \Rightarrow \phi = \frac{8\pi}{12} \Rightarrow \phi = \frac{2\pi}{3}$ rad

Ans. 5: (a)

Solution: $\because y = 10\sin\frac{2\pi}{5}(1800t - x)$ comparing this by $y = a\sin(\omega t - kx)$ we get

$\omega = \frac{3600\pi}{5} = 2\pi f \Rightarrow f = \frac{3600}{10} = 360\ Hz$ and $k = \frac{2\pi}{5} = 2\pi/\lambda$

$\Rightarrow \quad \frac{1}{T} = 360 \quad \Rightarrow \quad T = \frac{1}{360}s$

Ans. 6: (a)

Solution: Let the displacement in simple harmonic motion be represented as $x = A\sin\omega t$

For $x = \frac{A}{2}$, $\omega t_1 = \frac{\pi}{6}$ and For $x = A$, $\omega t_2' = \frac{\pi}{2}$

Time taken to go from $x = 0$ to $x = \frac{A}{2}$ is $t_1 = \frac{\pi}{6\omega}$

Time taken to go from $x = \frac{A}{2}$ to $x = A$ is $t_2 = t_2' - t_1 = \left(\frac{1}{\omega}\right)\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3\omega}$

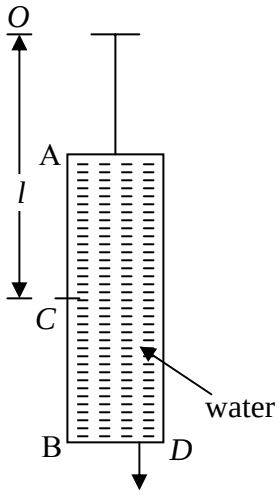
Hence $\frac{t_1}{t_2} = \frac{\pi/6\omega}{\pi/3\omega} = \frac{1}{2}$

Qualitatively, the velocity of the particle is maximum at its mean position and it decreases as it moves towards extreme of vibration. Hence, the particle will take lesser time to travel first half of the displacement as compared to the second-half displacement

Ans. 7: (d)

Solution: Let C be the middle point of a cylinder suspended from point O. As water slowly flows out of a hole at the bottom the water level in cylinder goes down so centre of gravity goes down thus effective length of the pendulum increases and by formula

$T = 2\pi\sqrt{\frac{l}{g}}$. Hence time period first increases.



After some time when water level comes below a certain level centre of gravity of whole system once again goes up, when whole water flows out c.g., again comes at C. Thus showing smaller T.

Ans. 8: (c)

Solution: In one complete cycle, the kinetic oscillates twice.

Ans. 9: (d)

Solution: We have $U = -\int F dx = -\int \left(-kx + ax^3\right)dx = k\frac{x^2}{2} - a\frac{x^4}{4} \Rightarrow U = 0$ at $x = 0$ and $x = \sqrt{\frac{2k}{a}}$.

The function U has a maximum value at $x = \sqrt{\frac{k}{a}}$ and function U is negative for $x > \sqrt{\frac{2k}{a}}$.

The above characteristics are shown by graph (d).

Ans. 10: (a)

Solution: Since $PE \propto x^2$, we will have $PE \propto \cos^2 \omega t$

The plots I and III represent above characteristics.

Ans. 11: (a)

Solution: The velocity of SHM is given as $v = \omega\sqrt{a^2 - y^2}$

The velocity of SHM at equilibrium position i.e. $y = 0$ is equal to $a\omega$

Thus at equilibrium $v = a\omega$

Since velocities of all three oscillators are equal hence by

$$\omega_1 A_1 = \omega_2 A_2 = \omega_3 A_3$$

Ans. 12: (c)

Ans. 13: (c)

Solution: If k is force constant and a is amplitude then total energy $= \frac{1}{2}ka^2$

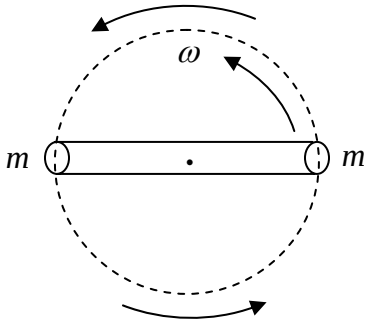
$$\Rightarrow 2 = \frac{1}{2} \times 400 \times a^2 \Rightarrow a^2 = \frac{4}{400} \Rightarrow a = \frac{1}{10} m$$

We know that $\omega^2 = \frac{k}{m} \Rightarrow \omega^2 = \frac{400}{1}(m = 1) \Rightarrow \omega = 20$

In SHM, acceleration $= -\omega^2 a = (20)^2 \times \left(\frac{1}{10}\right) = 40\ m/s^2$

Ans. 14: (a)

Solution: The reduced mass of the system



$$\mu = \frac{m \times m}{m+m} \Rightarrow \mu = \frac{m}{2}$$

If the angular velocity is ω then centripetal force $= T$

$$\Rightarrow T = \mu\omega^2 r \qquad \Rightarrow T = \frac{m}{2}\omega^2 \cdot 2l \because r = 2l, \mu = \frac{m}{2}$$

$$\Rightarrow \qquad \omega^2 = \frac{T}{ml} \Rightarrow \quad \omega = \sqrt{\frac{T}{ml}} \Rightarrow \quad 2\pi f = \sqrt{\frac{T}{ml}} \qquad \Rightarrow \text{ Frequency } f = \frac{1}{2\pi}\sqrt{\frac{T}{ml}}$$

Ans. 15: (d)

Solution: $T = 2\pi\sqrt{\frac{M}{R}} \Rightarrow \frac{T_2}{T_1} = \sqrt{\frac{2M}{M}} = \sqrt{2} \Rightarrow T_2 = \sqrt{2}T_1 = \sqrt{2}T$

Ans. 16: (b)

Solution: The force constant is inversely proportional to length. If the length l of the spring is cut into x and $l-x$ such that $x = 2(l-x)$ then $x = \frac{2}{3}$.

From the inverse relation, we can write $\frac{k_1}{k} = \frac{l}{x} = \frac{l}{2l/3} = \frac{3}{2}$

Ans. 17: (d)

Solution: First two spring are connected parallel hence displacement in both the spring will be equal say x then by Hook's law $F_1 = -k_1 x, \quad F_2 = -k_2 x \Rightarrow F_1 + F_2 = -(k_1 + k_2)x$

$$\Rightarrow F = -kx \Rightarrow k_{eq} = k_1 + k_2 \Rightarrow k_{eq} = 2k \quad \because k_1 = k_2 \Rightarrow T_1 = 2\pi\sqrt{\frac{m}{2k}}$$

In second case $T_2 = 2\pi\sqrt{\frac{m}{k}}$

In third case $k_{eq} = \frac{k}{2}$ then $T_3 = 2\pi\sqrt{\frac{2m}{k}}$

Hence $T_1 : T_2 : T_3 = \frac{1}{\sqrt{2}} : 1 : \sqrt{2}$

Ans. 18: (b) Solution: Change in gravitational energy = Energy stored in the spring

$$Mgx = \frac{1}{2}kx^2 \quad \text{or} \quad x = \frac{2Mg}{k}$$

Ans. 19: (b)

Solution: The time period of a simple pendulum is given by $T = 2\pi\sqrt{\frac{l}{g}}$

The acceleration of the lift is $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(1.0m/s^2\right)(2t) = 2.0m/s^2$

The new time period of the pendulum is $\frac{T'}{T} = \sqrt{\frac{g}{g'}} = \sqrt{\frac{10}{12}} = \sqrt{\frac{5}{6}}$

Ans. 20: (d) Solution: $T = 2\pi\sqrt{\frac{l}{g}}$

Now, new value of g is $g' = g + \frac{0.8}{100} \times g = g + 0.008g = g\left(1 + 0.008g\right)$

$\therefore$ New time period is $T' = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{g\left(1+0.008\right)}}$

Now $\frac{T'}{T} = \left(\frac{1}{1+0.008}\right)^{\frac{1}{2}} = \left(1+0.008\right)^{\frac{-1}{2}} = 1 - \frac{1}{2} \times 0.008 = 1 - 0.004$

$\Rightarrow \quad T' = T - 0.004T$

Now decrease in time $= T - T' = 0.004T$

$\therefore$ % decrease in time $= \frac{T - T'}{T} \times 100 = \frac{0.004T}{T} \times 100 = 0.4\%$

Ans. 21: (b) Solution: $T = 2\pi\sqrt{\frac{l}{g}}$

When pendulum acceleration in a horizontal direction, $g_{eff} = \sqrt{a^2 + g^2}$

$\therefore \quad T = 2\pi\sqrt{\frac{l}{\left(a^2 + g^2\right)^{1/2}}}$

Ans. 22: (c)

Solution: $T = 2\pi\sqrt{\frac{l}{g}}$

$$T' = 2\pi\sqrt{\frac{l}{g+a}} = 2\pi\sqrt{\frac{l}{g+\frac{g}{3}}} = 2\pi\sqrt{\frac{3l}{4g}} = \sqrt{\frac{3}{2}} \times 2\pi\sqrt{\frac{l}{g}} = \sqrt{\frac{3}{2}}T$$

Solution-MSQ

Ans. 23: (a), (b), (c) and (d)

Solution: In all given figures the acceleration is directly proportion negative of displacement, hence in all case motion is SHM.

Ans. 24: (a), (b), (d)

Solution: The total energy of SHM = $\frac{1}{2}mv_{\max}^2 = \frac{1}{2}m(\omega a)^2 = \frac{1}{2}ma^2\left(\frac{2\pi}{T}\right)^2 = 2\pi^2\frac{ma^2}{T^2}$

Hence, total energy depends on mass, amplitude and time period.

Ans. 25: (a), (b), (c) and (d)

Solution: (a) $v = \omega\sqrt{A^2 - x^2}$ at mean position $x = 0 \Rightarrow v = \omega$ (maximum)

(b) $\langle K.E\rangle = \frac{1}{u}m\omega^2A^2$ and $\langle P.E\rangle = \frac{1}{u}m\omega^2A^2 \Rightarrow \langle E\rangle = \frac{1}{2}m\omega^2A^2 = \text{constant}$

(c) $x = A\sin\omega t \Rightarrow v = \frac{dx}{dt} = A\omega\cos\omega t \Rightarrow a = \frac{d^2x}{dt^2} = -A\omega^2\sin\omega t = -\omega^2x$

At $x = A \qquad \Rightarrow a = a_{\max} = -\omega^2A$

(d) Restoring force is $F_r = -kx$. F_r is always direct towards mean position

Ans. 26: (b), (c) and (d)

Solution: The function will be S.H.M. if restoring force is proportional to displacement

(a) $x = \sin^2 \omega t$, $v = \dfrac{dx}{dt} = 2\omega \sin \omega t = \omega \sin 2\omega t \Rightarrow a = 2gw^2 \cos 2\omega t$

$F_r = ma = 2m\omega^2 \cos 2\omega t$ but not proportional to x .Thus this function is not S.H.M

(b) $x = \cos 2\omega t, v = -2\omega \sin 2\omega t, a = -4\omega^2 \cos 2\omega t \Rightarrow F = ma = -4\omega^2 x \Rightarrow F = -kx$.

Thus this function represents S.H.M.

(c) $x = \sin \omega t - \cos \omega t \Rightarrow v = \dfrac{dx}{dt} = \left[\cos \omega t + \sin \omega t\right]\omega \Rightarrow a = \dfrac{d^2x}{dt^2} = \left(-\sin \omega t + \cos \omega t\right)\omega^2$

$\therefore \quad F = ma = -m\omega^2 \left[\sin \omega t - \cos \omega t\right] = -m\omega^2 x = -kx$. Thus this function is S.H.M.

(d) $x = \sin \omega t + 2\cos \omega t \Rightarrow v = \dfrac{dx}{dt} = \left[\cos \omega t - 2\sin \omega t\right]\omega$

$a = \dfrac{d^2x}{dt^2} = \left[-\sin \omega t - 2\cos \omega t\right]\omega^2 = -\omega^2 \left[\sin \omega t + 2\cos \omega t\right] \Rightarrow F = ma = -m\omega^2 x = -kx$.

Thus function represent S.H.M.

Ans. 27: (b)

Solution: The restoring acceleration which results in simple harmonic motion of a pendulum acts tangential to the path.

Ans. 28: (b) and (d)

Solution: For spring pendulum, $T = 2\pi\sqrt{\dfrac{M}{K}}$

For simple pendulum, $T = 2\pi\sqrt{\dfrac{l}{g}}$

Thus time period of spring pendulum does not change whereas time period of simple pendulum increase as $g_{moon} < g_{earth}$

Thus correct options are (b) & (d)

Ans. 29: (c) and (d)

Solution: (a) Total energy $E = \frac{1}{2}kA^2$

If $A = 2A$ then $E = \frac{1}{2}k(2A)^2 = 4 \times \frac{1}{2}kA^2 = 4 \times$ energy

(b) $T = 2\pi\sqrt{\frac{M}{k}}$ does not depend on amplitude

(c) Maximum velocity $v_{max} = A\omega$ if $A = 2A$, $v'_{max} = 2A\omega = 2 \times v_{max}$

(d) Maximum acceleration $a'_{max} = 2(-A\omega^2) = 2a_{max}$

Thus only v_{max} & a_{max} doubled when A is doubled

Ans. 30: (a) and (b)

Solution:

(a) $x = a\sin\omega t \Rightarrow v = a\omega\cos\omega t = a\omega\sin\left(\omega t + \frac{\pi}{2}\right)$. Thus phase difference is $\frac{\pi}{2}$

(b) $x = a\sin\omega t \Rightarrow$ accel. $= -a\omega^2\sin\omega t = a\omega^2\sin(\omega t + \pi)$. Thus phase difference is π

(c) $(\text{accel.})_{max} = -a\omega^2$

(d) $v = \omega\sqrt{a^2 - x^2}$. Maximum velocity is not at $x = \pm A$ but at $x = 0$

Solution-NAT

Ans. 31: 2

Solution: The time period of the pendulum is given by $T = 2\pi\sqrt{\frac{l}{g}}$. The variation of g with the height above the earth is given by $g = \frac{GM_e}{(R_e + h)^2}$.

When $h = 0 \Rightarrow g_1 = \frac{GM_e}{R_e^2}$ and $h = R_e \Rightarrow g_2 = \frac{GM_e}{4R_e^2}$ $\Rightarrow \frac{T_2}{T_1} = \sqrt{\frac{g_1}{g_2}} = \sqrt{\frac{\frac{GM_e}{R_e^2}}{\frac{GM_e}{4R_e^2}}} = 2$

Ans. 32: 5028.5

Solution: The value of 'g' goes on decreasing as we move towards the centre of earth its value at a depth 'd' is given by $g_d = g\left(1 - \frac{d}{R}\right) = \frac{g}{R}(R - d) = \frac{gx}{R}$

where $x = R - d$ is the distance of body from the centre of earth.

Time period $T = 2\pi\left[\frac{\text{displacement}}{\text{Acceleration}}\right]^{\frac{1}{2}} = 2\pi\sqrt{\frac{x}{g_d}} \Rightarrow T = 2\pi\sqrt{\frac{x}{\frac{gx}{R}}} = 2\pi\sqrt{\frac{R}{g}}$

$\Rightarrow T = 2\pi\sqrt{\frac{6.4\times10^6}{10}} = 5028.5 \text{ sec}$ $\because R = 6400\ km = 6.4\times10^6\ m,\ g = 10 m/\sec^2$

Ans. 33: 107

Solution: $\therefore$ $T = 2\pi\sqrt{\frac{l}{g}} = 2\times3.14\sqrt{\frac{2}{9.8}} = 2.8 \text{ sec}$ here $l = 2\ m,\ g = 9.8\ m/s^2$

Thus number of complete oscillations in $2.8 \text{ sec} = 1$

Number of complete oscillation is $1\text{sec} = \frac{1}{2.8}$

$\therefore$ Number of complete oscillation in 5 minute (300 sec) is $= \frac{1}{2.8}\times300 = 107$

Ans. 34: 2

Solution: The time period of simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$

New length is $l' = l + \frac{4l}{100} = 1.04\, l \Rightarrow T' = 2\pi\sqrt{\frac{l'}{g}} = 2\pi\sqrt{\frac{1.04\, l}{g}}$

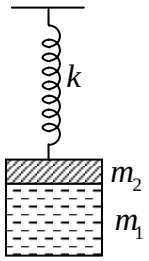
$\therefore \quad \frac{T'}{T} = \sqrt{1.04} = (1+0.04)^{\frac{1}{2}} = 1 + \frac{1}{2}\times 0.04 = 1 + 0.02 \Rightarrow T' = T + 0.02T$

Now increase in time period $= T' - T = 0.02T$

$\therefore$ % change in time period $\frac{T'-T}{T}\times 100 = \frac{0.02T}{T}\times 100 = 0.02\times 100 = 2\%$

Ans.35: 80

Solution: Let $y =$ extension in length of the spring when both m_1 and m_2 are suspended with it then $(m_1 + m_2)g = ky \Rightarrow y = \frac{(m_1+m_2)g}{k}$



Let $y' =$ extension in the length of the spring when only m_2 is suspended

$\therefore \quad m_2 g = ky' \Rightarrow y' = \frac{m_2 g}{k}$

$\Rightarrow y - y' = \frac{(m_1+m_2)g}{k} - \frac{m_2 g}{k} = \frac{m_1 g}{k}.$

Thus the amplitude of the oscillation $= \frac{m_1 g}{k} = \frac{1\times 10}{12.5} = 0.8\ m = 80\ cm$

Ans. 36: 1.6

Solution: Time period $T = 2\pi\sqrt{\frac{M}{K}} \Rightarrow 2 = 2\pi\sqrt{\frac{M}{K}}$ (For 1st case, $T = 2\,\text{sec}$)

also $3 = 2\pi\sqrt{\frac{M+2}{K}}$ (For 2nd case $T = 2+1 = 3\,\text{sec}$)

$\Rightarrow \frac{3}{2} = \sqrt{\frac{M+2}{M}} \Rightarrow \left(\frac{3}{2}\right)^2 = \frac{M+2}{M} \Rightarrow \frac{9}{4} = \frac{M+2}{M} \Rightarrow M = \frac{8}{5} = 1.6\,kg$

Ans. 37: 1

Solution: $y = a\sin\omega t = a\sin\frac{2\pi}{T}t$

For $y = \frac{a}{2}$, $\therefore \quad \frac{a}{2} = a\sin\frac{2\pi}{T}t \qquad \Rightarrow \frac{1}{2} = \sin\frac{2\pi}{T}t$

$\Rightarrow \sin\frac{\pi}{6} = \sin\frac{2\pi}{T}t \Rightarrow \frac{\pi}{6} = \sin\frac{2\pi}{T}t \Rightarrow t = \frac{T}{12} \Rightarrow t = \frac{12}{12} = 1\text{ sec}$ where $T = 12\text{sec}$

Ans. 38: 400

Solution: Consider spring constant of spring $= k$

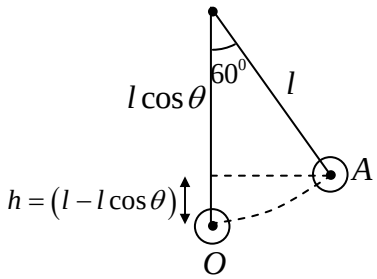
After cutting into 4 equal parts, the spring constant of each part is $k' = 4k$

$\therefore$ Spring constant of parallel combination of two parts $k' + k'$

$= 4k + 4k = 8k = 8\times 50 = 400\ N/m$

Ans. 39: 2

Solution: $K.E$ at the lowest position $= \frac{1}{2}mv^2 = \frac{1}{2}m\times(3)^2 = \frac{9}{2}m$.



When the length makes an angle $\theta(=60^\circ)$ to the vertical, the bob of the pendulum will have both $K.E$ and $P.E$.

If v is velocity of bob at this position and h is the height of the bob w.r.t. O. Then total energy of the bob $E = \frac{1}{2}mv^2 + mgh$

But $h = l - \cos\theta = l(1-\cos\theta) = 0.5(1-\cos 60^\circ) = 0.5\left(1-\frac{1}{2}\right) = \frac{1}{4}$

$\therefore \quad E = \frac{1}{2}mv^2 + m\times 10\times\frac{1}{4} = \frac{1}{2}mv^2 + \frac{5m}{2}$

According to law of conservation of energy

$\frac{1}{2}mv^2 + \frac{5m}{2} = \frac{9m}{2} \Rightarrow \frac{1}{2}mv^2 = \frac{9m}{2} - \frac{5m}{2} = 2m \Rightarrow v^2 = 4 \therefore \quad v = 2\ m/\text{sec}$

Ans. 40: 0.15

Solution: Maximum force, $F = ma_{\max}$

Now $a_{\max} = \omega^2 a = \left(\frac{2\pi}{T}\right)^2 a = \frac{4\pi^2}{T^2} a = \frac{4\pi^2 \times 25}{\pi^2} \times 0.3 = 30 \ m/\sec^2$

$\therefore \ F = 5 \times 10^{-3} \times 30 = 0.15 \ N$

Ans. 41: 10.5

Solution: $T = 2\pi\sqrt{\frac{l}{g}}$

$\therefore \quad T' = 2\pi\sqrt{\frac{l(1+0.21)}{g}} \qquad \because l' = l + \frac{21}{100} l = l + 0.21l = l(1+0.21)$

Now $\frac{T'}{T} = \sqrt{1+0.21} = 1 + \frac{1}{2}(0.21) \Rightarrow T' = T + \frac{1}{2}(0.21)T$

$\therefore$ Increase in time period $= T' - T = \frac{0.21}{2} T = 0.105T$

$\therefore$ Percentage increase $= \frac{T' - T}{T} \times 100 = 0.105 \times 100 = 10.5\%$

2. Damped and Forced Oscillators

2.1 Damped Harmonic Oscillation

The majority of the oscillatory systems in everyday life suffer some sort of irreversible energy loss due to frictional or viscous heat generation while they are oscillating. Their amplitude of oscillation dies away with time. Such oscillation is called damped harmonic oscillation.

Consider the mass-spring system, body of mass m attached to spring with spring constant k is released from position x_0 (measured from equilibrium position) with velocity v; the mass is subject to a *frictional damping force* which opposes its motion, and is directly proportional to its instantaneous velocity $F_{res} = -bv$

The quantity v is a positive constant, whose value depends on the properties of the material providing the resistance. The minus sign indicates that the force resists the motion, so it is directed *opposite* to the velocity.

The total force on the body is the sum of the restoring force $F = -kx$, and the resistive force F_{res}: $F_{net} = F + F_{res} = -kx - bv$

Using Newton second law $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \Rightarrow \frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0$

Here, we define the constant γ such that, $2\gamma = \frac{b}{m}$ called damping constant and

$\omega_0 = \sqrt{k/m}$ is the natural frequency of undamped oscillator.

Assuming a solution of the form $x(t) = Ce^{\alpha t}$

On differentiation, we get $\left\{(\alpha)^2 + 2\alpha\gamma + \omega_0^2\right\} Ce^{\alpha t} = 0$

The solution (for ω) from the quadratic formula is: $\alpha = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2}$

$$\Rightarrow \alpha = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Thus the solution is $x(t) = Ae^{-\gamma t}e^{\pm\sqrt{\gamma^2-\omega_0^2}t} = Ae^{-\gamma t}e^{\pm j\omega' t}$ where $\omega' = \sqrt{\omega_0^2 - \gamma^2}$.

If we consider the quantity under the square root sign, we see that there are three possibilities.

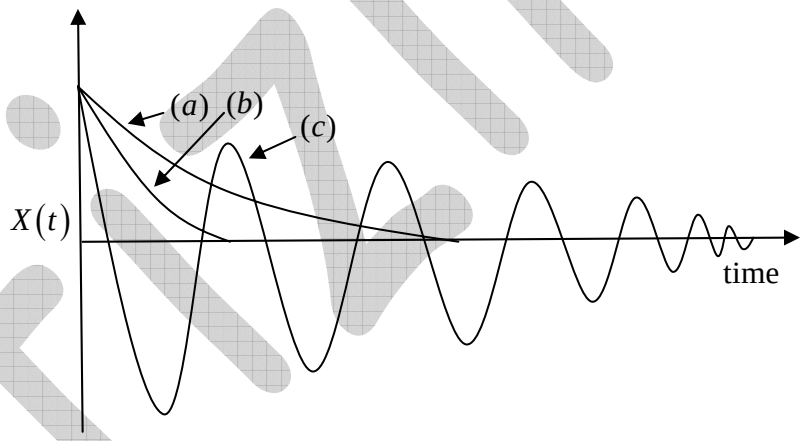
Case 1: Overdamped Case: If the damping coefficient is large, then $\gamma > \omega_0$ and ω' will be imaginary. Hence $x(t)$ will be a negative exponential function. It is shown in figure (a).

Case 2: Critically Damped Case: If $\omega_0 = \gamma$, then the square root vanishes. In this case, the solution is again a negative exponential function which goes to zero quicker than the overdamped case, as shown in figure (b).

Case 3: Underdamped Case: If $\omega_0 > \gamma$ then the quantity under the square root is positive and we have a real number for ω'. The solution for 'x' is then

$$x(t) = Ae^{-\gamma t}\sin(\omega' t + \phi)$$

This is the solution of the damped harmonic oscillator. The oscillatory motion is shown in figure (c).



Time Period: of the damped harmonic oscillator is: $T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}}$

This shows that due to damping the time period slightly increased.

Logarithmic Decrement: This measure the rate at which the amplitude decay

$$\lambda = \gamma T' = \frac{bT'}{2m}$$

Mean Life time (τ_m)**:** It is the time taken for the amplitude to decay to $\frac{1}{e}$ of the initial value. When $t = \tau_m = \frac{1}{\gamma}$, Amplitude $= \frac{A}{e}$ $\left(\because Ae^{-\gamma t}\right)$.

Energy of the Damped Oscillator

(i) Kinetic energy (K)

The displacement of a damped harmonic oscillator is $x(t) = Ae^{-\gamma t}\sin(\omega' t + \phi)$

The instantaneous velocity is

$$u = \frac{dx}{dt} = Ae^{-\gamma t}\left[-\gamma\sin(\omega' t + \phi) + \omega'\cos(\omega' t + \phi)\right] \approx Ae^{-\gamma t}\omega'\cos(\omega' t + \phi)$$

The approximation is done as $\gamma << \omega_0$

Thus, kinetic energy is $K = \frac{1}{2}mu^2 = \frac{1}{2}mA^2e^{-2\gamma t}\omega'^2\cos^2(\omega' t + \phi)$

(ii) Potential energy (U)

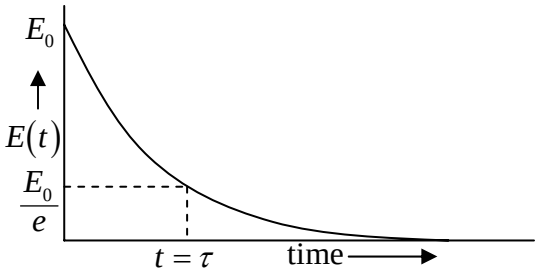
$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2e^{-2\gamma t}\sin^2(\omega' t + \phi)$$

(iii) Total energy (E)

$$E = K + U = \frac{1}{2}mA^2e^{-2\gamma t}\omega'^2\cos^2(\omega' t + \phi) + \frac{1}{2}kA^2e^{-2\gamma t}\sin^2(\omega' t + \phi)$$

$$\Rightarrow E = \frac{1}{2}kA^2e^{-2\gamma t} \qquad \text{Since } \gamma << \omega_0 \quad \text{thus } \omega' = \sqrt{\omega_0^2 - \gamma^2} \approx \omega_0 = \sqrt{\frac{k}{m}}$$

This shows that the energy of the oscillator dcreases with time, the exponential decay of energy is show below.



Power Dissipation: It is the rate at which the energy is lost

$$P = -\frac{dE}{dt} = 2\gamma\left(\frac{1}{2}kA^2e^{-2\gamma t}\right) = 2\gamma E$$

Relaxation Time: It is the time taken for the total energy to decay to $\frac{1}{e}$ of its initial value E_0. If τ is the relaxation time, then at $t = \tau$, we shall have $E = \frac{E_0}{\tau}$.

Thus $\frac{E_0}{\tau} = E_0e^{-2\gamma\tau}$. This gives relaxation time $\tau = \frac{1}{2\gamma}$

Thus we can also write power dissipated as $P = \frac{E}{\tau}$

and energy can be expressed as $E(t) = E_0e^{-\frac{t}{\tau}}$.

Quality Factor (*Q*): It is defined as the 2π times the ratio of the energy stored in the system to the energy lost per period.

$$Q = 2\pi\frac{\text{energy stored in system}}{\text{energy loss per period}}$$

Energy stored in the system is E while the energy loss per period is PT, thus

$$Q = 2\pi\frac{E}{PT} = 2\pi\frac{E}{\left(\frac{E}{\tau}\right)T} = \frac{2\pi\tau}{T} = \omega_0\tau$$

This shows that, lower the damping, higher the value of Q.

Now energy can be written in term of Q as $E(t) = E_0e^{-\frac{\omega_0 t}{Q}}$

It means that Q is related to the number of oscillation over which the energy fall to $\frac{1}{e}$ of its original value E_0, which is also called the relaxation time. This happens in time, $t = \tau$, where τ is given by

$$\frac{\omega_0 \tau}{Q} = 1 \quad \Rightarrow \tau = \frac{Q}{\omega_0} = \frac{TQ}{2\pi}$$

In one period (T) number of oscillation is $=1$

In time τ the number of oscillation $= n$, then $n = \frac{\tau}{T} = \frac{Q}{2\pi}$

Thus the energy falls to $\frac{1}{e}$ of its original value after $n = \frac{Q}{2\pi}$ cycle of free oscillation.

Relation between relaxation time, mean time and Quality factor:

Let N is the number of oscillation in time $\tau_m = \frac{1}{\gamma}$

While n is the number of oscillation in time $\tau = \frac{1}{2\gamma}$

Thus, relation between mean and relaxation time is $\tau_m = 2\tau$

In one period (T) number of oscillation is $=1$

In time τ_m the number of oscillation $= N$

$$N = \frac{\tau_m}{T} = \frac{2\tau}{T} = 2\frac{Q}{2\pi} \quad \text{or} \quad N = 2\frac{Q}{2\pi} = 2n$$

2.2 Forced Oscillations and Resonance

When a body is made to oscillate under external periodic force, then in the beginning the body tries to oscillate with its natural frequency, but very soon these oscillation dies out and the body oscillate with the frequency of the applied force. Such oscillation is called forced oscillation.

Example:

1) Person swinging in a swing without anyone pushing → **Free or damped oscillation**

2) Someone pushes the swing periodically → **Forced or driven oscillations**

Natural angular frequency (ω_0) of the system when system oscillates freely after a sudden disturbance.

External frequency (p) of the system is the angular frequency of the external driving force causing the driven oscillations.

Resonance

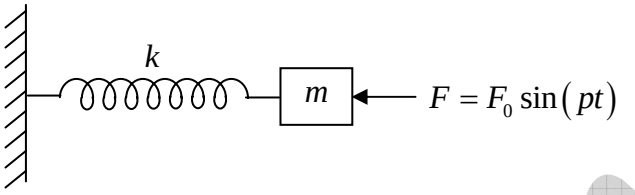
When the frequency of the external force is same as the natural frequency of the body, the amplitude of the oscillation is maximum. This phenomenon is called resonance.

Transient Effect

In the presence of external periodic force, initially body tries to oscillate with its natural frequency, while external force tries to impose its own frequency. Thus there is a tussel between external force and the body during which the amplitude rises and falls alternatively. This is the transient effect which soon dies out and body start oscillating with external frequency.

Equation of Forced Oscillation

Consider a mass *m* oscillating under external periodic force *F*. let *x* be the displacement at any instant. The forces acting on the mass are



(i) Restoring force proportional to the displacement. This is written as $F = -kx$

(ii) A frictional force proportional to the velocity. This is written as $F_{res} = -b\frac{dx}{dt}$

(iii) An external periodic force, which is $F = F_0 \sin(pt)$

Thus the total force on the body is $F = -kx - b\frac{dx}{dt} + F_0 \sin(pt)$

Using newton's third law, we can write $\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = f_0 \sin(pt)$

where $2\gamma = \frac{b}{m}$, $\omega_0^2 = \frac{k}{m}$ and $f_0 = \frac{F_0}{m}$

This is the differential equation of the forced harmonic oscillator.

General solution of the differential equation: Once the transient effect dies out, body starts oscillating with frequency of the external force. So we can assume a solution of differential equation as $x = A\sin(pt - \theta)$

Differentiating and putting in differential equation we obtain the value of amplitude A

$$A = \frac{f_0}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}}$$

phase difference θ between the displacement and the driving force is $\tan\theta = \frac{2\gamma p}{\omega_0^2 - p^2}$

Thus solution of the forced oscillator becomes $x = \frac{f_0}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}}\sin(pt - \theta)$

Amplitude Resonance

The amplitude A of the forced oscillator depends on the constant f_0 and p of the driving force and the constant ω_0 and γ of the oscillator. At certain driving frequency amplitude A becomes maximum that is called amplitude resonance. Three different cases arise

Case 1: At very low driving frequency $(p << \omega_0)$. The amplitude A turns to be

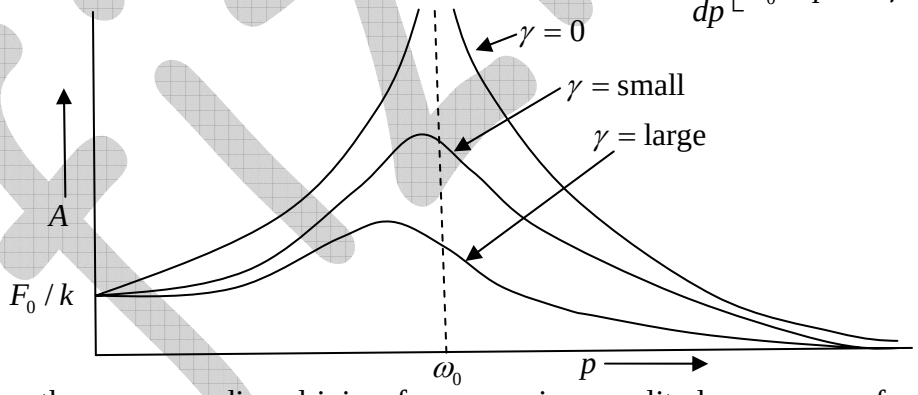
$$A = \frac{f_0}{\omega_0^2} \cong \frac{F_0}{k}$$

This shows that the amplitude depends only upon force constant and independent of mass, damping constant and driving frequency.

Case 2: At very high frequency $(p >> \omega_0)$, we get $A = \frac{f_0}{p^2} \cong \frac{F_0}{mp^2}$

This shows that amplitude now depends upon mass and driving frequency.

Case 3: Amplitude will be maximum when the denominator $\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}$ in A is minimum. So that its first derivative coefficient will be zero $\frac{d}{dp}\left[\omega_0^2 - p^2 + 4\gamma^2 p^2\right] = 0$



This gives the corresponding driving frequency i.e. amplitude resonance frequency at which A is maximum is $p = \sqrt{\left(\omega_0^2 - 2\gamma^2\right)}$.

At this frequency the amplitude is $A_{max} = \frac{f_0}{2\gamma\sqrt{\gamma^2 + p^2}} = \frac{f_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$

This shows that maximum amplitude depends upon the damping constant γ, smaller the damping larger is the amplitude. When, $\gamma = 0$, $A_{max} \to \infty$.

Dependence of the phase of displacement on the frequency of driving force

In steady state condition, the force equation and displacement equation is

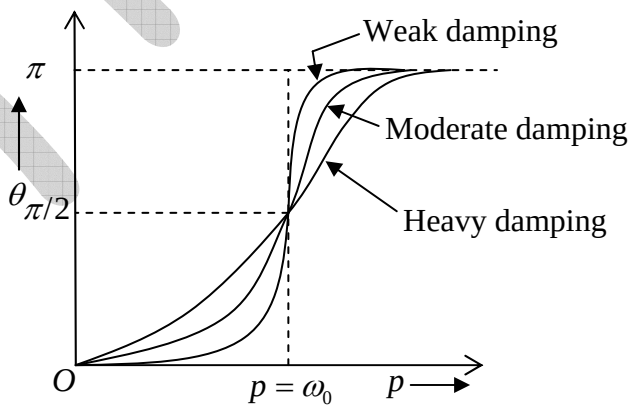
$F = F_0 \sin(pt)$ and $x = A\sin(pt - \theta)$, the phase factor is $\tan\theta = \dfrac{2\gamma p}{\omega_0^2 - p^2}$

This indicates that the displacement of the forced oscillation lags behind the driving force $F_0 \sin(pt)$ by an angle θ. The phase difference θ depends upon the damping γ and also on the difference between natural and driving frequency.

Case 1: If $p << \omega_0$, than $\tan\theta$ is positive and θ lies between 0 and $\dfrac{\pi}{2}$. In this condition (when $\gamma \approx 0$), θ is nearly zero i.e. displacement is in phase with driving force

Case 2: If $p = \omega_0$, than $\tan\theta$ is infinite and $\theta = \dfrac{\pi}{2}$. This is the resonance condition, displacement always lag behind the force by $\dfrac{\pi}{2}$. It means that at resonance the displacement is minimum when the driving force is maximum and vice versa.

Case 3: If $p >> \omega_0$, than $\tan\theta$ is negative and θ lies between $\dfrac{\pi}{2}$ and π. In this condition, θ is nearly π i.e. displacement almost opposite in phase with the driving force. The variation of θ with driving frequency p is given below.



Velocity Resonance

Velocity of the body also depends on the constant f_0 and p of the driving force and the constant ω_0 and γ of the oscillator. At certain driving frequency velocity amplitude becomes maximum that is called velocity resonance.

The instantaneous velocity of the body is

$$u = \frac{dx}{dt} = \frac{f_0 p}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}} \cos\left(pt - \theta\right)$$

The velocity will be maximum when $\cos\left(pt - \theta\right) = 1$. The maximum value is known as "velocity amplitude" u_0. Thus

$$u_0 = \frac{f_0 p}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}}$$

Three different cases arise.

Case 1: At very low driving frequency $\left(p << \omega_0\right)$. The velocity amplitude u_0 turns to be

$$u_0 = \frac{f_0 p}{\omega_0^2} = \frac{F_0 p}{k}$$

This indicates that the velocity amplitude depends on the spring constant k.

Case 2: At very high frequency $\left(p >> \omega_0\right)$, we get $u_0 = \frac{f_0}{p} = \frac{F_0}{mp}$

It shows that velocity amplitude depends on mass as well as driving frequency.

Case 3: At certain frequency the velocity amplitude becomes maximum, that frequency is called the velocity resonance frequency. The velocity amplitude can be written as

$$u_0 = \frac{f_0 p}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}} = \frac{f_0}{\sqrt{\left(\frac{\omega_0^2 - p^2}{p}\right)^2 + 4\gamma^2}}$$

The velocity amplitude u_0 will be maximum when the denominator $\sqrt{\left(\frac{\omega_0^2 - p^2}{p}\right)^2 + 4\gamma^2}$ is minimum i.e. when

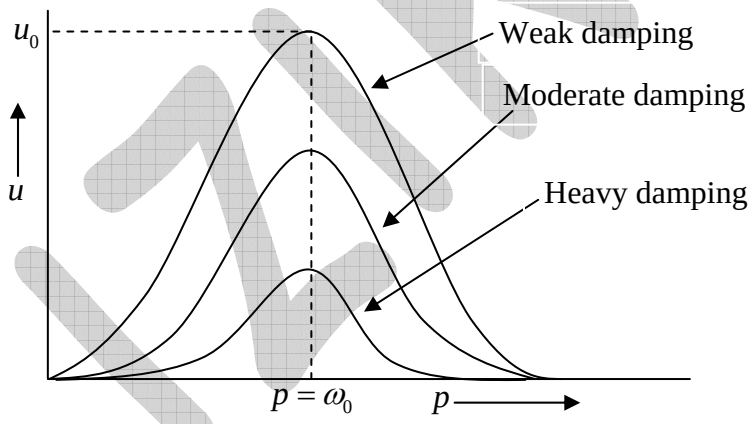
$$\left(\frac{\omega_0^2 - p^2}{p}\right) = 0 \Rightarrow p = \omega_0$$

This shows that the velocity resonance irrespective of damping value, always occurs when driving frequency is equal to the natural undamped frequency of the body.

The velocity amplitude at the resonance is $u_0 = \frac{f_0}{2\gamma} = \frac{F_0}{b}$

This shows that velocity amplitude at resonance only depends on the damping constant.

Dependence of the velocity of a forced oscillator on the driving frequency is shown as below



Dependence of the phase of velocity on the frequency of driving force:

In steady state condition, the force equation and velocity equation is

$$F = F_0 \sin(pt)$$

$$u = u_0 \cos(pt - \theta) = u_0 \sin\left(pt - \left(\theta - \frac{\pi}{2}\right)\right) = u_0 \sin(pt - \phi) \Rightarrow \phi = \theta - \frac{\pi}{2}$$

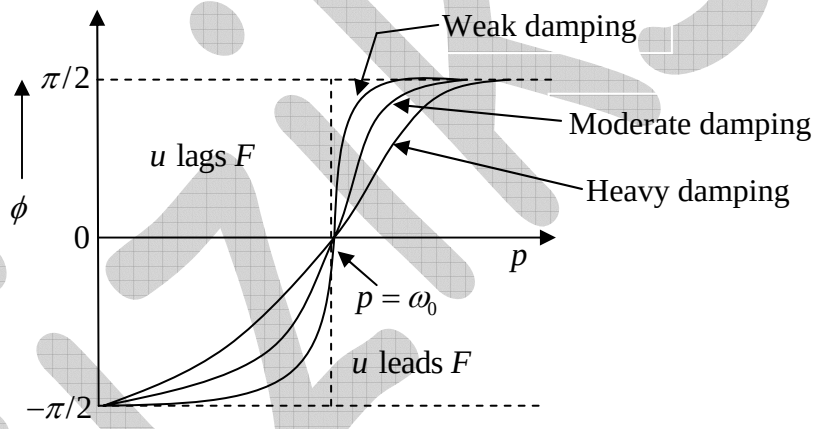
$$\Rightarrow \tan\phi = \cot\theta = \frac{\omega_0^2 - p^2}{2\gamma p} \qquad \because \tan\theta = \frac{2\gamma p}{\omega_0^2 - p^2}$$

Case 1: If $p << \omega_0$, than $\tan\theta$ is positive and θ lies between 0 and $\frac{\pi}{2}$. In this condition (when $\gamma \approx 0$), θ is nearly zero. Thus $\phi = -\frac{\pi}{2}$, this means velocity leads the driving force.

Case 2: If $p = \omega_0$, than $\tan\theta$ is infinite and $\theta = \frac{\pi}{2}$, and $\phi = 0$, this means that at resonance, the velocity is always in phase with the driving force.

Case 3: If $p >> \omega_0$, than $\tan\theta$ is negative and θ lies between $\frac{\pi}{2}$ and π, so that the ϕ is positive. This means that velocity lags behind the driving force.

The dependence of phase of velocity of the body on the frequency of the driving force is shown below.



Average power absorbed by oscillator (supplied by the driving force)

An oscillator absorbed energy from the driving force which is dissipated in doing work against the damping force present.

The instantaneous power P (i.e. rate at which work is done) absorbed by the oscillator is equal to energy per unit time.

$$P_{in} = \frac{\text{energy}}{\text{time}} = \frac{F.dx}{dt} = F.\frac{dx}{dt}$$

Thus P_{in} is equal to the product of the instantaneous driving force and the instantaneous velocity.

Thus

$$P_{in} = F_0 \sin(pt)\frac{dx}{dt} = \frac{mf_0^2 p}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}} \sin(pt).\cos(pt-\phi)$$

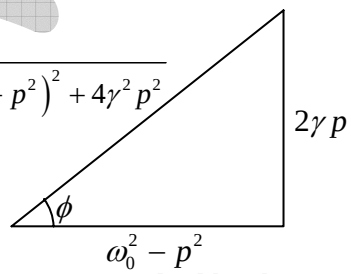
$$= \frac{mf_0^2 p}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}} \left(\sin pt \cos pt \cos\phi\right) + \sin^2 pt \sin\phi$$

Now the average power absorbed is $\langle P_{in} \rangle = \frac{mf_0^2 p}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}} \left(\frac{1}{2}\sin\phi\right)$

Note: the average values of the periodic function for the one period $T = \frac{2\pi}{p}$ is

$$\frac{1}{T}\int_0^T \sin pt \cos pt\, dt = 0 \quad \text{and} \quad \frac{1}{T}\int_0^T \sin p^2 t dt = \frac{1}{2}$$

Since, $\tan\theta = \frac{2\gamma p}{\omega_0^2 - p^2}$, thus using the vector model



$$\sin\phi = \frac{2\gamma p}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}}$$

Therefore, the average power absorbed by the oscillator (average power supplied by the driving force) is

$$\langle P_{in} \rangle = \frac{mf_0^2 \gamma p^2}{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}$$

The power supplied by the driving force is not stored in the system, but is dissipated as work done in moving the system against the force of friction.

Power Dissipated through frictional force

The instantaneous power dissipated through friction is given by

$P_{dis} = |\text{instantaneous frictional force}| \times |\text{instantaneous velocity}|$

$$= b\frac{dx}{dt} \times \frac{dx}{dt} = 2m\gamma\left(\frac{dx}{dt}\right)^2 = 2m\gamma \frac{f_0^2 p^2}{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}\cos^2\left(pt - \phi\right)$$

Now the average of $\cos^2\left(pt - \phi\right)$ for one full period is $\frac{1}{2}$. Therefore the average power dissipated is $\left\langle P_{dis}\right\rangle = \frac{mf_0^2 \gamma p^2}{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}$

Thus $\left\langle P_{in}\right\rangle = \left\langle P_{dis}\right\rangle$

This shows that in the steady state the average power supplied by the driving force is equal to the average power dissipated by the frictional force.

Note: The instantaneous input power is not equal to the instantaneous power dissipated. Therefore at any instant of time the power stored in the oscillator is not constant.

Maximum power absorption

The average power can also be written as

$$\left\langle P_{in}\right\rangle = \frac{mf_0^2 \gamma}{\left(\frac{\omega_0^2 - p^2}{p}\right)^2 + 4\gamma^2}$$

$\left\langle P_{in}\right\rangle$ will be maximum when the denominator $\left(\frac{\omega_0^2 - p^2}{p}\right)^2 + 4\gamma^2$ is a minimum, this occurs when $\omega_0^2 - p^2 = 0$ or $p = \omega_0$

this is the condition of velocity resonance. Hence the power transferred from the driving force is maximum at the frequency of velocity resonance. The maximum power is

$$\left\langle P_{in}\right\rangle_{\text{maximum}} = \frac{mf_0^2}{4\gamma}$$

Sharpness of Resonance and Bandwidth

Sharpness of Resonance

The rapidity with which the power falls from its resonant value with change in a driving frequency is known as Sharpness of resonance.

The average power is maximum at certain frequency. As the driving frequency deviates either way from its resonant value, the power falls from its maximum value. If the fall in power with change in driving frequency from the resonant value is large, the resonance is said to be sharp, on the other hand, the fall is small, the resonant is said to be flat.

It is measured by the ratio of the resonant frequency ω_0 to the difference of two frequencies ω_1 and ω_2 at which the power falls to half of the resonant value.

$$\text{Sharpness of resonance} = \frac{\omega_0}{\omega_2 - \omega_1}$$

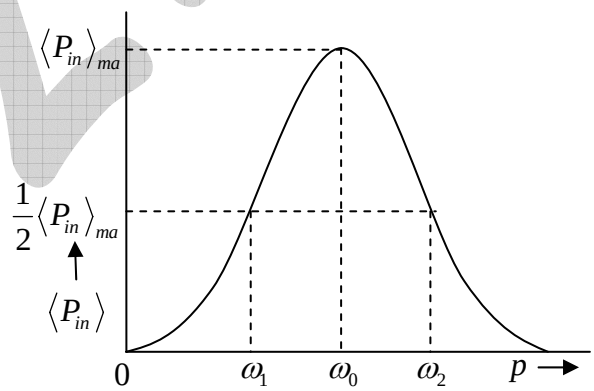
Bandwidth of Resonance

The difference in values of the driving frequency, at which the average power absorbed drops to half its maximum value, is called the band width of the resonance.

$$\text{Bandwidth} = \omega_2 - \omega_1$$

The average power absorbed is

$$\langle P_{in} \rangle = \frac{m f_0^{\,2} \gamma p^2}{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}$$



The maximum average power is

$$\langle P_{in} \rangle_{\text{maximum}} = \frac{m f_0^{\,2}}{4\gamma}$$

The variation of the average power with driving frequency is shown in above figure also shows the half power frequencies.

The value of p at which the power goes half the maximum value obtained as

$$\langle P_{in} \rangle = \frac{1}{2}\langle P_{in} \rangle_{\text{maximum}} \quad \Rightarrow \quad \frac{m f_0^{\,2} \gamma 2 p^2}{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2} = \frac{1}{2}\frac{m f_0^{\,2}}{4\gamma}$$

or $\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2 = 8\gamma^2 p^2 \quad \Rightarrow \quad \left(\omega_0^2 - p^2\right)^2 = 4\gamma^2 p^2$

or $\omega_0^2 - p^2 = \pm 2\gamma p \quad \Rightarrow \quad p^2 = \omega_0^2 \pm 2\gamma p$

These are two quadratic equations in p,

$$p^2 + 2\gamma p - \omega_0^2 = 0 \qquad \text{and} \qquad p^2 - 2\gamma p - \omega_0^2 = 0$$

each has two roots one positive and other negative. Since negative frequency are not allowed, thus the allowed positive roots are

$p_1\left(=\omega_1\right) = -\gamma + \sqrt{\gamma^2 + 2\gamma\omega_0^2}$ and $p_2\left(=\omega_2\right) = \gamma + \sqrt{\gamma^2 + 2\gamma\omega_0^2}$

The frequency difference between tow half power points i.e. bandwidth is

$$\text{Bandwidth} = \omega_2 - \omega_1 = 2\gamma = \frac{1}{\tau}$$

and sharpness of resonance $= \dfrac{\omega_0}{\omega_2 - \omega_1} = \omega_0\tau$

It indicates that smaller the bandwidth, sharper is the resonance.

Quality Factor

The quality factor is defined as $Q = \dfrac{\omega_0}{\omega_2 - \omega_1} = \dfrac{\text{Resonance frequency}}{\text{bandwidth}} = \dfrac{\omega_0}{2\gamma} = \omega_0\tau$

Quality factor also defined as

$$Q = 2\pi \frac{\text{average energy stored in one period}}{\text{average energy lost in one period}} = 2\pi \frac{E}{\langle P_{dis}\rangle T}$$

Where $\langle P_{dis}\rangle$ is the average power dissipated and T is the time period of oscillation. Thus $\langle P_{dis}\rangle \times T$ is the average energy lost in one period. On solving for energy we get the following expression of the quality factor.

$$Q = \frac{p^2 + \omega_0^2}{4\gamma p} = \frac{1}{2}\left(\frac{p^2 + \omega_0^2}{p^2}\right)(p\tau)$$

This is the exact expression for the quality factor of forced oscillator.

Near resonance, $p = \omega_0$, so we get $Q = \omega_0\tau$

MCQ (Multiple Choice Questions)

Q1. A mass m attached to a spring is oscillating in water. If the spring constant is k and frictional force is $R\dot{\psi}$, which one of the following is correct?

(a) $m\ddot{\psi} = R\dot{\psi} + k\psi$ (b) $m\ddot{\psi} = R\dot{\psi} - k\psi$

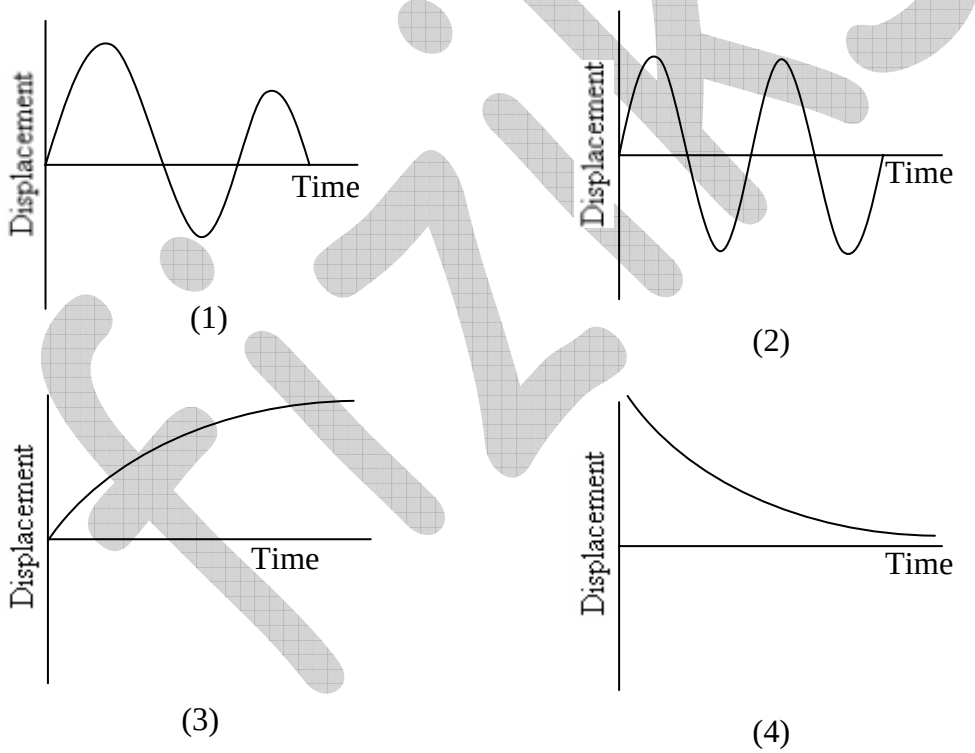
(c) $m\ddot{\psi} = -R\dot{\psi} + k\psi$ (d) $m\ddot{\psi} = -R\dot{\psi} - k\psi$

Q2. A lightly damped oscillator have a characteristic frequency ω. When operating frequency $\Omega << \omega$, the response of the oscillator is controlled by:

(a) damping coefficient (b) inertia of the mass

(c) oscillator frequency (d) spring constant

Q3.



Which of the above figure(s) represent(s) damped simple harmonic motion?

(a) Figure 1 alone (b) Figure 2 alone

(c) Figure 4 alone (d) Figure 3 and 4

Q4. Which one of the following expressions correctly represents forced oscillation?

(a) $\frac{d^2y}{dt^2} + \omega^2 y = 0$ (b) $\frac{d^2y}{dt^2} = 0$

(c) $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + \omega^2 y = 0$ (d) $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + \omega^2 y = F\sin pt$

Q5. A spring oscillating in water is acted upon by an external force $B\cos\omega t$. With the passage of time the frequency of the spring tends to be:

(a) greater than ω (b) less than ω

(c) equal to ω (d) decreasing exponentially

Q6. A damped simple harmonic oscillator of frequency f_1 is constantly driven by an external periodic force of frequency f_2. At the steady state, the oscillator frequency will be:

(a) f_1 (b) f_2 (c) $f_1 - f_2$ (d) $\frac{f_1 + f_2}{2}$

Q7. In the case of forced simple harmonic vibrations, the body generally vibrates with:

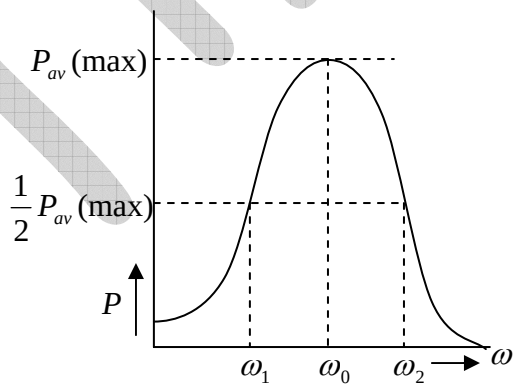
(a) Its natural frequency of vibration and its amplitude is small

(b) its natural frequency of vibration but its amplitude is large

(c) the frequency of the external force with a small amplitude

(d) the frequency of the external force with a large amplitude

Q8.



The above graph shows the average power P_{av} against the frequency of the driving force acting on an oscillator.

Consider the following statements:

(1) The sharpness of the peak at resonance is determined by the damping constant.

(2) The sharpness of resonance is defined by $\frac{\omega_2 - \omega_1}{\omega_0}$

(3) The peak occurs at the frequency of velocity resonance when the power absorbed by the oscillator from the driving force is maximum.

Which of these statements are correct?

(a) 1 and 2 (b) 2 and 3 (c) 1 and 3 (d) 1, 2 and 3

Q9. The maximum amplitude in the case of a forced oscillator occurs at the

(a) natural frequency of the oscillator

(b) frequency of the force

(c) frequency greater than the natural frequency of the oscillator

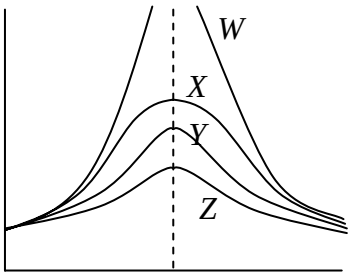
(d) frequency less than the natural frequency of the oscillator

Q10. If the differential equation given by $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + \omega^2 y = F_0 \sin pt$ describe the oscillatory motion of body in a dissipative medium under the influence of a periodic force, then the state of maximum amplitude of the oscillation is a measure of:

(a) free vibration (b) damped vibration

(c) forced vibration (d) resonance

Q11. The four curves shown in the given figure represent the variation of amplitude (A) with driving frequency (ω) for different values of damping constant. The curve which represents zero damping is the curve labeled:



(a) W (b) X

(c) Y (d) Z

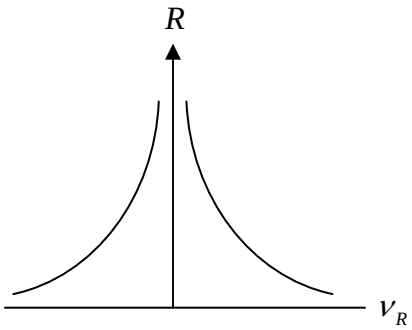
Q12. In the case of a forced vibration, the resonance wave becomes very sharp when the:

(a) damping force is small (b) restoring force is small

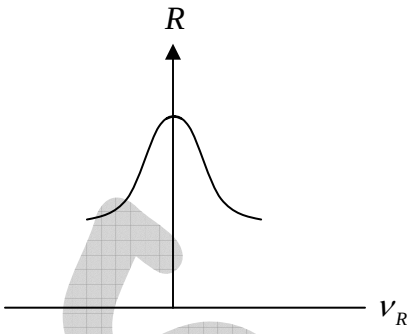
(c) applied oscillatory force is small (d) quality factor is small

Q13. The frequency response curve of a damped forced oscillator will be of the form (R = amplitude, ν_R = frequency):

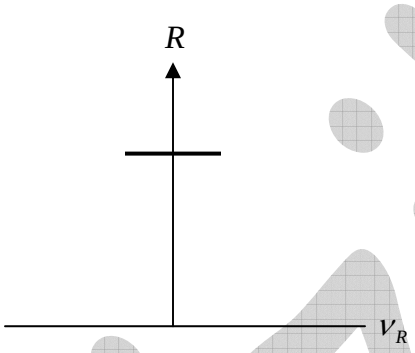
(a)



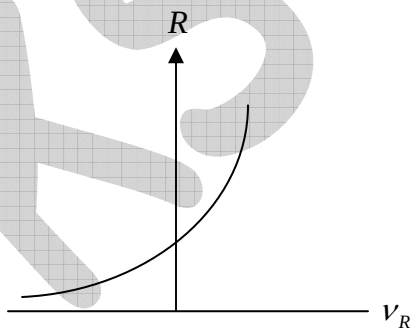
(b)



(c)



(d)



MSQ (Multiple Select Questions)

Q14. The equation of motion of an object of mass m attached with a spring of spring constant K is oscillating in medium of damping constant b is written as

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+Kx=0$$

if ω_0 is the natural angular frequency of the free oscillator thus which of the following are true?

(a) The critical damping occurs when $b=\sqrt{2mK}$

(b) Non-oscillatory, a periodic motion occurs when $b>2m\omega_0$

(c) Oscillations about equilibrium with an exponentially decaying amplitude occurs when $b<2m\omega_0$

(d) Non-oscillatory and faster return to equilibriums occurs when $b=2\sqrt{mK}$

Q15. A massless spring suspended from a rigid support, carries a flat disc of mass $200\ g$ at its lower end. It is observed that the system oscillates with a frequency of $20\ Hz$ and amplitude of the damped oscillation reduces to half of its undamped value in one minute. Which of the following statements are correct?

(a) Time period of oscillation decreases as amplitude decreases with time

(b) The damping constant b is $4.6\times10^{-3}\ Nsm^{-1}$

(c) The time in which amplitude decays to $\frac{1}{e}$ of itial value is 8.6 sec

(d) The time in which energy decays to $\frac{1}{e}$ of its itial value is 4.3 sec

Q16. A massless spring of spring constant $20\ N/m$ is suspended from a rigid support and carries a mass $0.2kg$ at its lower end. It is observed that the system performs damped oscillatory motion and its energy decays to $\frac{1}{e}$ of initial value in 50 sec. Which of the following statements are correct?

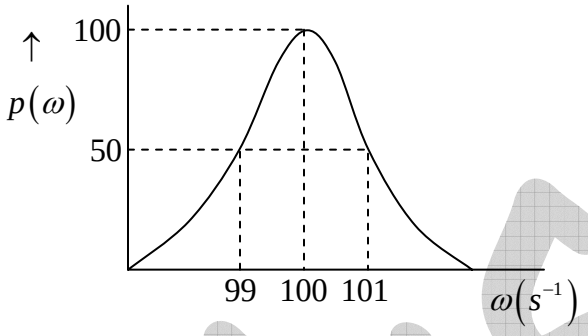
(a) The time in which amplitude decays to $\frac{1}{e}$ of initial value is 100 sec

(b) Damping constant b is $8\times10^{-3}\ Nsm^{-1}$

(c) Quality factor Q is 500

(d) Natural frequency of oscillation ω_0 is $10\ rad/\text{sec}$

Q17. The graph shows the power resonance curve of a certain mechanical system which is driven by a force of constant magnitude but variable angular frequency ω of the following statements are correct?



(a) The resonance frequency ω_0 is 100 $rad/\sec$

(b) The full width at half maxima (FWHM) is $2\sec^{-1}$

(c) The Quality factor Q is 100

(d) The energy decays to $\frac{1}{e}$ of initial value in the absence of driving force in 0.5 sec

fiziks

NAT (Numerical Answer Type)

Q18. A pendulum with a length 1.0 m is released from an initial angle of 15^0. After 1000 sec, its amplitude has been reduced by friction to 5.5^0. If amplitude $A = A_0 e^{-\gamma t}$ then the value of γ is ……..$\sec^{-1}$

Q19. A 10.6 kg object oscillates at the end of a vertical spring that has a spring constant of 2.05×10^4 N/m. The effect of air resistance is represented by damping co-efficient $b = 3N.s/m$. The frequency of the damped is …… Hz

Q20. A 2.0 kg object attached to a spring moves without friction and is driven by an external force given by $F = (3.0\,N)\sin(2\pi t)$. If the force constant of the spring is 20 N/m, then amplitude of the motion is ……..…. cm

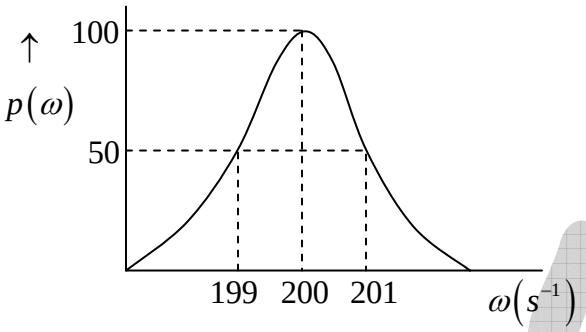
Q21. A weight of $40N$ is suspended from a spring that has a force constant of $20\frac{N}{m}$. The system is undamped and is subjected to a harmonic driving force of frequency 10 Hz resulting in a forced- motion amplitude of 2 cm. The maximum value of the driving force is……..…. N

Q22. The Q-value of an underdamped harmonic oscillator of frequency 480 Hz is 80000. The number of oscillation does it take in time in which its amplitude decays to $\frac{1}{e}$ of its initial value is ……..….

Q23. The energy of a piano string of frequency 256 Hz reduces to half its initial values in 2 sec. The Q-vlaue of the string is ……

Q24. The quality factor of a sonometer wire of frequency 500 Hz is 5000. The time in which its energy decays to $\frac{1}{e}$ of its initial value is …….sec

Q25. The graph shows the power resonance curve of a certain mechanical system which is drives by a force of constant magnitude but variable frequency ω



The quality factor of the system is ……

Q26. An object of mass $2\ kg$ hangs from a spring of negligible mass and spring constant of $800\ N/m$. The spring is extended by $2.5\ cm$ when the object is attached. The top end of the spring is oscillated up and down in SHM with an amplitude of $2\ mm$. The damping constant is $0.5\ \text{sec}^{-1}$. The amplitude of forced oscillations at $p = \omega_0$ is …… cm

Solution-MCQ (Multiple Choice Questions)

Ans. 1: (d)

Solution: $-R\dot{\psi}$ is frictional force and is applied against the acceleration and if k is spring constant then $-k\psi$ represent restoring arise due to extension in string and act in opposite direction of motion. The total force acting on mass *m* is

$$F = F_1 + F_2 = -R\dot{\psi} - k\psi \Rightarrow m\ddot{\psi} = -R\dot{\psi} - k\psi$$

Ans. 2: (d)

Q2. If the angular frequency of vibration (ω') of a damped simple harmonic oscillator is related to ω_0

$$\omega_0^2 - \omega'^2 = 10^{-6}\omega_0^2$$

where ω_0 is the angular frequency of its vibration when there is no damping, the Q-factor will be equal to:

(a) 500 (b) 1,000 (c) 5,000 (d) 10000

Ans. 2: (a)

Quality factor is defined as $Q = \dfrac{\omega_0}{2\gamma}$ where $\omega' = \sqrt{\omega_0^2 - \gamma^2} \Rightarrow \omega_0^2 - \omega'^2 = \gamma^2$

Thus $\gamma^2 = 10^{-6}\omega_0^2 \Rightarrow \gamma = 10^{-3}\omega_0 \Rightarrow Q = \dfrac{\omega_0}{2\gamma} = \dfrac{\omega_0}{2\times 10^{-3}\omega_0} = 500$

Ans. 3: (a)

Solution: In damped simple harmonic motion the amplitude dies exponentially.

Ans. 4: (d)

Solution: If mass of a particle is acted up by a periodic force.

Given as force $= F_0 \sin pt$, then its equation is given as

$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + cy = F_0 \sin \omega t$$

$$\Rightarrow \quad \frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{c}{m}y = \frac{F_0}{m}\sin pt \Rightarrow \frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + \omega^2 y = F\sin pt$$

Ans. 5: (c)

Solution: When a spring oscillates in water under periodic force $F = B\cos\omega t$, the oscillation is like forced oscillation and so after some time the spring oscillate with frequency equal to that of the force.

Ans. 6: (b)

Solution: When an external periodic force is switched on a harmonic oscillator, the motion of the oscillator in the beginning is irregular because the oscillator tries to oscillate with its own frequency where as the driving force gets success in imposing its own frequency. In this state, called the steady state, the oscillator oscillates with constant amplitude and with frequency equal to that of the driving force.

Ans. 7: (c)

Solution: In forced SHM the body oscillates with the frequency of external force and its amplitude is small.

Ans. 8: (c)

Solution: The quality factor Q measure sharpness and $Q = \dfrac{\omega_0}{\omega_2 - \omega_1}$

Sharpness = $Q = \dfrac{\omega_0}{\omega_2 - \omega_1}$

Hence statement (2) is wrong

Ans. 9: (d)

Solution: The equation of forced oscillation is given as

$$\ddot{x} + 2\gamma x + \omega_0^2 x = f_0 \sin pt$$

Amplitude of the motion is given as $A = \dfrac{f_0}{\left[\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2\right]^{\frac{1}{2}}}$

This becomes maximum when $p^2 = \left(\omega_0^2 - 2\gamma^2\right)$

$\Rightarrow$ The maximum amplitude occurs when $p < \omega_0$

Ans. 10: (d) Solution: In forced SHM the state of maximum amplitude of oscillation is a measure of resonance.

Ans. 11: (a)

Solution: $\because A_{\max} = \dfrac{f_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$. Thus as damping decreases the response curve becomes sharper and so its amplitude increases gradually. Thus when damping is zero, amplitude becomes infinite. Thus the curve *W* represents the response curve having zero damping.

Ans. 12: (a)

Solution: If m be mass of a particle, then forced vibration equation is given as

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + \frac{c}{m}x = F_0 \sin pt \qquad \text{-----(i)}$$

where F_0 is magnitude of applied force of frequency p.

Dividing equation (i) by *m*, we get

$\dfrac{d^2x}{dt^2} + \dfrac{r}{m}\dfrac{dx}{dt} + \dfrac{c}{m}x = \dfrac{F_0}{m}\sin pt$ or $\dfrac{d^2x}{dt^2} + 2k\dfrac{dx}{dt} + \omega_0^2 x = f_0 \sin pt$ where $f_0 = \dfrac{F_0}{m}$.

When steady state has been attained the oscillator has settled down to oscillate with forcing frequency $\dfrac{p}{2\pi}$ and a constant amplitude.

Let solution of equation (i) be given as $x = A\sin(pt - \theta)$

where θ is phase difference between the applied force and displacement of the oscillator.

Solution can be written as

$$A = \frac{f_0}{\left[\left(\omega_0^2 - p^2\right)^2 + 4k^2p^2\right]^{1/2}} \text{ where } \theta = \tan^{-1}\frac{2kp}{\left(\omega_0^2 - p^2\right)}$$

At resonance frequency amplitude is given as $A_{\max} = \dfrac{f_0}{2k\left(p^2 + k^2\right)^{1/2}}$

Sharpness is a measure of the rate of fall of amplitude from its maximum value at resonance frequency on either side of it. The sharper the fall in amplitude, the sharper the resonance.

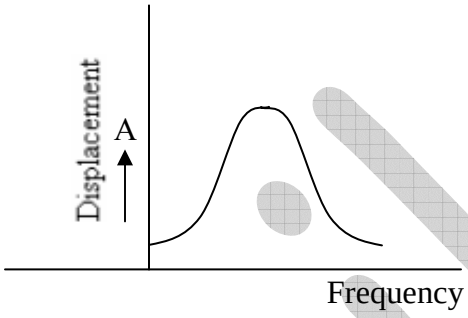
Ans. 13: (b)

Solution: The differential equation of forced oscillation is given as

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt$$

solving this the amplitude is given as $A = \dfrac{f_0}{\left[\left(\omega_0^2 - p^2\right) + \left(2kp\right)^2\right]^{1/2}}$

The curve given for this amplitude is given as



Solution-MSQ (Multiple Select Questions)

Ans. 14: (b), (c) and (d)

Solution: The damped frequency of the oscillator is $\omega = \sqrt{\omega_0^2 - \dfrac{b^2}{4m^2}}$

(i) Critical damping occurs when $b = 2m\omega_0 = 2\sqrt{mK}$. This provides for the faster return to equilibrium

(ii) Under damping (or light damping) occurs, when $b < 2m\omega_0$

This entails oscillation about equilibrium with exponentially decaying amplitude

(iii) Over damping (or heavy damping) occurs when $b > 2m\omega_0$.

This gives non-oscillatory, a periodic motion with a monatomic approach to equilibrium at late times, which slower than critical damping.

Ans. 15: (b) and (c)

Solution: (a) Time period of damped oscillation is independent of its amplitude

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}}$$

(b) The amplitude of the damped oscillator at an instant t is given by $A = A_0 e^{-\gamma t}$.

$$\Rightarrow \frac{1}{2} = e^{-\gamma \times 60} \qquad \because \frac{A}{A_0} = \frac{1}{2} \text{ when } t = 1 \text{ minute} = 60 \text{ sec}$$

$$e^{60\gamma} = 2 \qquad \text{or } \gamma = \frac{2.303 \times 0.301}{60} = 0.116 s^{-1}$$

$\therefore$ The damping constant b is

$$b = 2\gamma m = 2 \times 0.116 s^{-1} \times 200 \times 10^{-3}\ kg = 4.6 \times 10^{-3}\ Nsm^{-1}$$

(c) The time in which amplitude decay to $\frac{1}{e}$ of itial value is $\tau = \frac{1}{\gamma} = 8.6$ sec

(d) The time in which energy decays to $\frac{1}{e}$ of itial value is $\Gamma = 2\tau = 17.24$ sec

Ans. 16: (b), (c) and (d)

Solution: Given $\Gamma = 50 \sec$

(a) $\tau = \frac{\Gamma}{2} = \frac{50}{2} = 25$ sec

(b) $b = \frac{2m}{\tau} = \frac{2 \times 0.2}{25} = 8 \times 10^{-3}\ Nsm^{-1}$

(c) $Q = \omega\Gamma$, $\omega = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \left(\frac{1}{\tau}\right)^2} = \sqrt{(10)^2 - \left(\frac{1}{25}\right)^2} = \sqrt{99.9} = 9.99$

$\omega \cong 10\, rad/\sec \qquad \therefore \quad Q = 10 \times 50 = 500$

(d) $\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{20}{0.2}} = 10\, rad/\sec$

Ans. 17: (a), (b) and (d)

Solution: (a) The maximum value of the mean power input occurs when the angular frequency ω of driving force is equal to the resonant angular frequency ω of the oscillator

$\therefore \quad \omega = \omega_0 = 100\, rad/\sec$

(b) $FWHM = \Delta\omega = 2\sec^{-1}$

(c) $Q = \dfrac{\omega_0}{\Delta\omega} = \dfrac{100}{101-99} = \dfrac{100}{2} = 50$

(d) $E = E_0 e^{-(\Delta\omega)t} = E_0 e^{-2t}$, when $t = \dfrac{1}{2} = 0.5\sec$, energy E decays to $\dfrac{1}{e}$ its initial value

Solution-NAT (Numerical Answer Type)

Ans. 18: 0.001

Solution: $\theta_0 = 15^0$ and $\theta(t = 1000) = 5.5^0$

Now, $A = A_0 e^{-\gamma t} \Rightarrow \dfrac{A}{A_0} = \dfrac{A_0 e^{-\gamma 1000}}{A_0} = \dfrac{5.5}{15} \Rightarrow \ln\left(\dfrac{5.5}{15}\right) = -1000\gamma \therefore \quad \gamma = 0.001 \sec^{-1}$

Ans. 19: 7

Solution: The frequency of undamped oscillator is $\omega_0 = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{2.05\times10^4\, N/m}{10.6\, kg}} = 44\ Hz$

The frequency of damped oscillation is

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = \sqrt{(44)^2 - \left(\frac{3}{2\times1.06}\right)^2} = \sqrt{1933.96 - 0.02} = 44\ \sec^{-1}$$

$$\therefore \quad f = \frac{\omega}{2\pi} = \frac{44}{2\pi} = 7Hz$$

Ans. 20: 5.09

Solution: Given $F=(3.0N)\sin(2\pi t)$ and $k=20\dfrac{N}{m}$

$\therefore \quad \omega=\dfrac{2\pi}{T}=2\pi \ rad/\sec$ also $\omega_0=\sqrt{\dfrac{k}{m}}=\sqrt{\dfrac{20}{2}}=3.10 \ rad\sec$

The equation for the amplitude of a driven oscillator, with $b=0$, gives

$$A=\frac{F_0}{m}\left(\omega^2-\omega_0^2\right)^{-1}=\frac{3}{2}\left[4\pi^2-(3.16)^2\right]^{-1} \qquad \Rightarrow A=0.0509 \ m=5.09 \ cm$$

Ans. 21: 318

Solution: Amplitude of a driven oscillator with no damping $A=\dfrac{\dfrac{F_0}{m}}{\sqrt{\left(\omega^2-\omega_0^2\right)^2}}$

where $\omega=2\pi f=20\pi \ \sec^{-1}$, $\omega_0^2=\dfrac{k}{m}=\dfrac{200}{\left(\dfrac{40}{9.8}\right)}=49 \ \sec^{-2}$

$$\therefore \quad F_0=mA\left(\omega^2-\omega_0^2\right)=\left(\frac{40}{9.8}\right)\left(2\times10^{-2}\right)(3950-49)=318 \ N \Rightarrow F_0=318 \ N$$

Ans. 22: 25464.8

Solution: if τ_m is the mean time in which amplitude decays to $\dfrac{1}{e}$ of initial value. Then the number of oscillation in time τ_m is $N=\dfrac{Q}{\pi}=\dfrac{80000}{\pi}=25464.8$

Ans. 23: 4641

Solution: The average energy of the oscillation is $E(t)=E_0e^{\frac{-\omega t}{Q_0}}$

$$\Rightarrow \quad \frac{1}{2}=e^{\frac{-2\omega_0}{Q_0}} \quad \text{in } t=2\sec,\ E(t)=\frac{E_0}{2} \Rightarrow \quad e^{\frac{2\omega_0}{Q_0}}=2$$

$$\Rightarrow \quad \frac{2\omega_0}{Q_0}=\ln 2 \Rightarrow \quad Q_0=\frac{2\omega_0}{\ln 2}=\frac{2\times2\pi\nu_0}{\ln 2}=\frac{4\pi\nu_0}{\ln 2} \therefore \quad Q_0=\frac{4\times\pi\times256}{\ln 2}=\frac{3216.99}{0.693}=4641$$

Ans. 24: 1.6

Solution: The average energy of oscillation is $E(t) = E_0\ e^{\frac{-\omega_0 t}{Q_0}}$. It means that Q is related to the number of oscillation over which the energy fall to $\frac{1}{e}$ of its initial value E_0. This happens in time $t = \tau$ where $\Rightarrow \quad \frac{\omega_0 \tau}{Q} = 1 \Rightarrow \quad \tau = \frac{Q}{\omega_0} = \frac{Q}{2\pi\nu_0}$

$\therefore \quad T = \frac{5000}{2\pi(500)} = \frac{10}{2\pi} = \frac{5}{\pi} = 1.59 = 1.6 \text{ sec} \qquad \because Q = 5000,\ \nu_0 = 500\ Hz$

Ans. 25: 100

Solution: quality factor is defined as $Q = \frac{\omega_0}{\Delta\omega}$, where $\omega_0 = 200\ rad/\text{sec}$

and $\Delta\omega = 201 - 199 = 2\ rad/\text{sec} \Rightarrow Q = \frac{200}{2} = 100$

Ans. 26: 4

Solution: The amplitude of forced oscillation is $A = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - p^2) + 4\gamma^2\omega^2}}$

At $p = \omega_0$, we get $A = \frac{\frac{F_0}{m}}{2\gamma\omega_0}$. To evaluate A, we need to know F_0, the amplitude of the driving force and γ which measures the damping of system.

$F_0 = \text{spring constant} \times \text{displacement amplitude} = (800\ N/m)(2\times10^{-3}\ m) = 1.6\ N$

where $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{800\ N/m}{2}} \Rightarrow \quad \omega_0 = 20\ rad/\text{sec}$

$A = \frac{\frac{1.6}{2}}{2\times0.5\times20} = \frac{0.8}{20} = 0,04\ m \quad A = 4\ cm$

3. Superposition of Two or more Simple Harmonic Oscillations

3.1 Superposition of Two Collinear Harmonic Oscillations

3.1.1 Addition of Two S.H.M having Equal Frequencies

Consider two SHMs of equal frequencies but of different amplitudes and phase constants acting on a particle in the x-direction.

The displacement x_1 and x_2 of the two SHM of frequency ω is

$$x_1 = A_1 \sin(\omega t + \phi_1) \text{ and } x_2 = A_2 \sin(\omega t + \phi_2)$$

where A_1 and A_2 are the amplitude and ϕ_1 and ϕ_2 the initial phases of the two motions.

The resultant displacement x at any instant t is

$$x = x_1 + x_2 = A_1 \sin(\omega t + \phi_1) = A_2 \sin(\omega t + \phi_2)$$

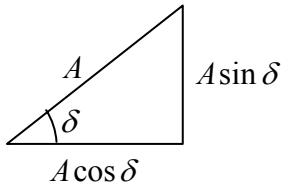
$$\Rightarrow x = A_2 \sin(\omega t + \phi_2) = A_1(\sin \omega t \cos\phi_1 + \cos \omega t \sin\phi_1) + A_2(\sin \omega t \cos\phi_2 + \cos \omega t \sin\phi_2)$$

$$\Rightarrow x = \sin \omega t (A_1 \cos\phi_1 + A_2 \cos\phi_2) + \cos \omega t (A_1 \sin\phi_1 + A_2 \sin\phi_2)$$

Now Let

$A_1 \cos\phi_1 + A_2 \cos\phi_2 = A\cos\delta$ and $A_1 \sin\phi_1 + A_2 \sin\phi_2 = A\sin\delta$

where A and δ are constant to be determined.



$$\Rightarrow x = A\sin(\omega t + \delta)$$

This shows that the resultant motion is SHM with angular frequency ω, the same as that of the individual SHMs.

The resultant motion has amplitude A and a phase constant δ

where $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2)}$ and $\tan\delta = \dfrac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}$

Conclusion: superposition of collinear SHM is also a SHM of the same frequency but different amplitude and phase constant

Case-I: (Maximum Amplitudes)

When the phase difference between the two individual motions is zero or any integral multiple of 2π i.e. $\phi_1 - \phi_2 = 2n\pi$ $\quad n = (0,1,2,3...)$

Then $A = A_1 + A_2$ (Resultant amplitude is sum of the amplitudes of individual motions)

Case-I: (Minimum Amplitudes)

When $\phi_1 - \phi_2 = (2n+1)\pi$, $n = (0,1,2,3...)$ this gives $A = A_1 - A_2$

If $A_1 = A_2$ then $A = 0$, i.e particle at rest.

3.1.2 Addition of Two S.H.M having Different Frequencies

Consider two SHM of equal amplitude but different frequencies

$$x_1 = A_1 \sin \omega_1 t \text{ and } x_2 = A_1 \sin \omega_2 t$$

The resultant displacement is

$$x = x_1 + x_2 = A_1 (\sin \omega_1 t + \sin \omega_2 t) = 2A_1 \sin\left(\frac{\omega_1 + \omega_2}{2}\right) t \cos\left(\frac{\omega_1 - \omega_2}{2}\right) t$$

$$x = A \sin\left(\frac{\omega_1 + \omega_2}{2}\right) t$$

This represent a periodic motion) of amplitude $A = 2A_1 \cos\left(\frac{\omega_1 - \omega_2}{2}\right) t$

$$\Rightarrow A = 2A_1 \cos 2\pi\left(\frac{\nu_1 - \nu_2}{2}\right) t \qquad \because \omega_1 = 2\pi\nu_1 \text{ and } \omega_1 = 2\pi\nu_2$$

Thus resultant amplitude of motion varies periodically between $\pm 2A_1$ and zero.

The amplitude A is maximum when $\cos 2\pi\left(\frac{\nu_1 - \nu_2}{2}\right) t = \pm 1$

$$\left(\frac{\nu_1 - \nu_2}{2}\right) t = 0, \pi, 2\pi, 3\pi, 4\pi.... \quad \text{or} \quad t = 0, \frac{1}{\nu_1 - \nu_2}, \frac{2}{\nu_1 - \nu_2}, \frac{3}{\nu_1 - \nu_2}.....$$

Hence the time interval between two consecutive maxima is $T_b = \frac{1}{\nu_1 - \nu_2}$.

The frequency ν_b of maxima is $= \nu_1 - \nu_2$

The amplitude A is minimum when $\cos 2\pi\left(\frac{\nu_1 - \nu_2}{2}\right) t = 0$

or $$2\pi\left(\frac{\nu_1 - \nu_2}{2}\right) t = \left(n + \frac{1}{2}\right)\pi \qquad (n = 0,1,2,3....)$$

or $t = \dfrac{\left(n+\dfrac{1}{2}\right)}{\nu_1 - \nu_2} = \dfrac{1}{2(\nu_1-\nu_2)}, \dfrac{3}{2(\nu_1-\nu_2)}, \dfrac{3}{2(\nu_1-\nu_2)} \ldots\ldots$

Thus the time interval between two consecutive minima is $T_b = \dfrac{1}{(\nu_1-\nu_2)}$

Hence the frequency of minima is also $(\nu_1-\nu_2)$.

One maximum of amplitude followed by a minimum is called a beat. The time period T_b between the successive beats is called the beat period $T_b = \dfrac{1}{(\nu_1-\nu_2)}$

and beat frequency $\nu_b = \dfrac{1}{T_b} = (\nu_1-\nu_2)$.

3.2 Superposition of Two Perpendicular Harmonic Oscillations

3.2.1 Addition of Two SHM having Equal Frequencies

Let us consider two perpendicular SHM one along x-axis and other along y-axis with amplitude A_1 & A_2

$x = A_1 \sin(\omega t + \delta)$ and $y = A_2 \sin \omega t$ where δ is phase constant.

$$\Rightarrow \frac{x}{A_1} = \sin \omega t \cos\delta + \cos \omega t \sin\delta \Rightarrow \frac{x}{A_1} = \frac{y}{A_2}\cos\delta + \sqrt{1-\frac{y^2}{A_2^2}}\sin\delta$$

$$\because \sin \omega t = \frac{y}{A_2} \text{ and } \cos \omega t = \sqrt{1-\sin^2 \omega t} = \sqrt{1-\frac{y^2}{A_2^2}}$$

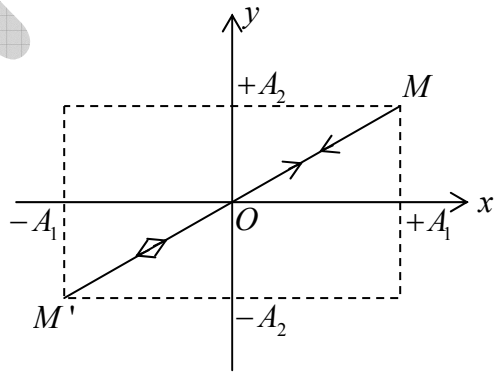
$$\Rightarrow \left(\frac{x}{A_1} - \frac{y}{A_2}\cos\delta\right) = \sqrt{1-\frac{y^2}{A_2^2}}\sin\delta \Rightarrow \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1A_2}\cos\delta = \sin^2\delta$$

This is the general equation of ellipse whose axes are inclined to the co-ordinate axes.

Let us consider few cases

(i) $\delta = 0 \Rightarrow \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1A_2} = 0$

or $\left(y - \frac{A_2}{A_1}x\right)^2 = 0$ or $y = \frac{A_2}{A_1}x$



This is the equation of straight line having a positive slope $\frac{A_2}{A_1}$ and passing through the origin.

Motion description: $x = A_1 \sin \omega t$ and $y = A_2 \sin \omega t$

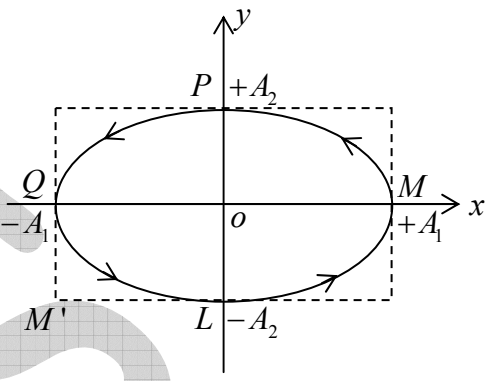
At time $t = 0$, particle is at O, at $t = \frac{T}{4}(\omega t = \pi/2)$, particle is at M.

At $t = \frac{T}{2}(\omega t = \pi)$, particle is at O, at $t = \frac{3T}{4}\left(\omega t = \frac{3\pi}{2}\right)$, particle is at M'.

and at $t = T(\omega t = 2\pi)$, particle is at O

Such vibration is called linearly polarized vibration.

(ii) $\delta = \dfrac{\pi}{2} \Rightarrow \dfrac{x^2}{A_1^2} + \dfrac{y^2}{A_2^2} = 1$ which is equation of ellipse as shown in figure .The particle moves in an elliptical path. The direction of its motion can be obtained as



$x = A_1 \cos \omega t$ and $y = A_2 \sin \omega t$

At $t = 0, (\omega t = 0) : x = A_1,\ y = 0$ i.e. particle is at M

At $t = \dfrac{T}{4} (\omega t = \pi/2) : x = 0, y = +A_2$ i.e. particle is at P

At $t = \dfrac{T}{2} (\omega t = \pi) : x = -A_1, y = 0$ i.e. particle is at Q

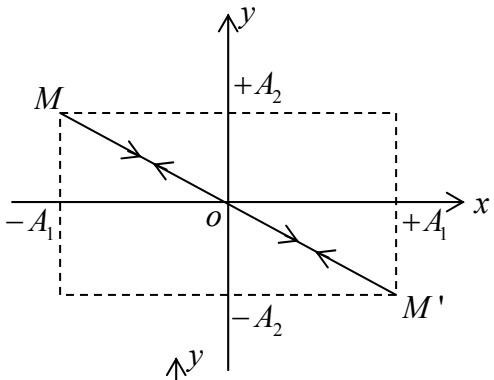
At $t = \dfrac{3T}{4} \left(\omega t = \dfrac{3\pi}{2} \right) : x = 0, y = -A_2$ i.e. particle is at L

At $t = T (\omega t = 2\pi) : x = +A_1, y = 0$ i.e. particle is at M

Thus the particle traces out an ellipse in the anti-clockwise direction. Such vibration is called LEFT-HANDED elliptically polarized vibration.

In addition, If $A_1 = A_2 = A$, the motion become circular $\left(x^2 + y^2 = A^2 \right)$ with radius A.

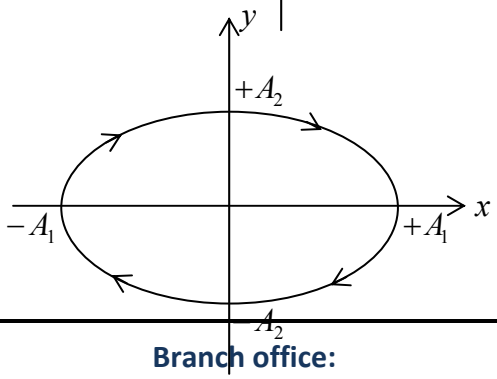
(iii) $\delta = \pi \Rightarrow \dfrac{x^2}{A_1^2} + \dfrac{y^2}{A_2^2} + \dfrac{2xy}{A_1 A_2} = 0$



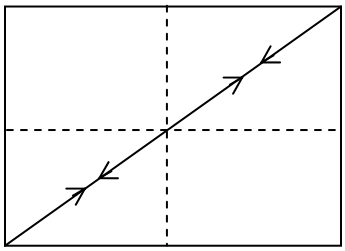
or $\left(\dfrac{x}{A_1} + \dfrac{y}{A_2} \right)^2 = 0$ or $y = \dfrac{-A_2}{A_1} x$

This represents a equation of straight line, having negative slope $\dfrac{-A_2}{A_1}$ & passing through the origin.

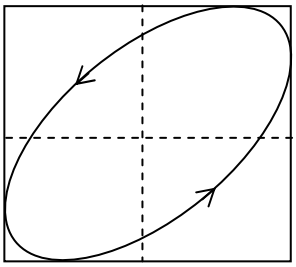
(iv) $\delta = \dfrac{3\pi}{2} \quad \Rightarrow \dfrac{x^2}{A_1^2} + \dfrac{y^2}{A_2^2} = 1$



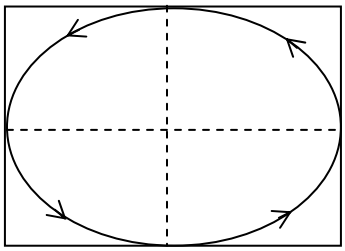
Such vibration is called RIGHT-HANDED elliptically polarized vibration.



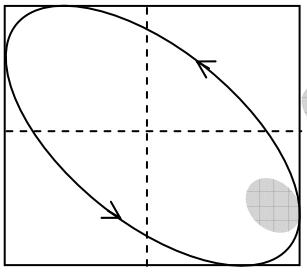
$\delta = 0$



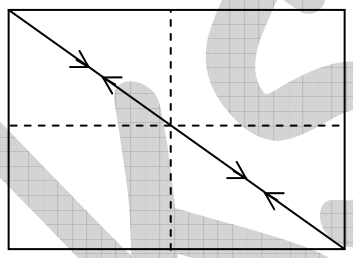
$\delta = \frac{\pi}{4}$



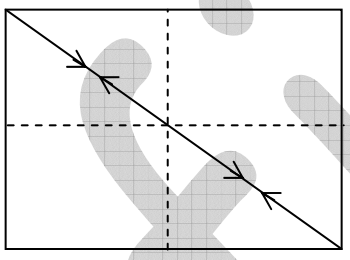
$\delta = \frac{\pi}{2}$



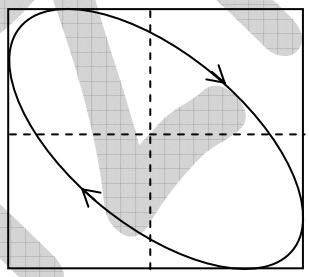
$\delta = \frac{3\pi}{4}$



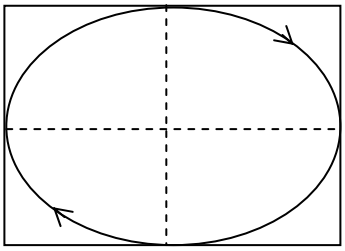
$\delta = \pi$



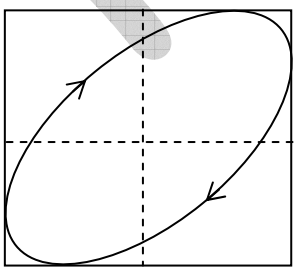
$\delta = \pi$



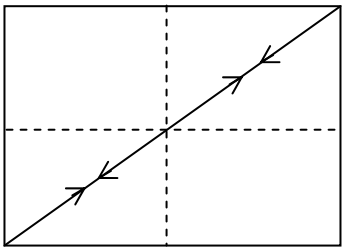
$\delta = \frac{5\pi}{4}$



$\delta = \frac{3\pi}{2}$



$\delta = \frac{7\pi}{4}$



$\delta = 2\pi$

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3.2.2 Addition of Two SHM having Different Frequencies (Lissajous Figures)

(1) Frequencies in the Ratio of 2:1

Let a particle is subjected to two mutually perpendicular SHM having frequencies $\omega_1 : \omega_2 = 2:1$

$$x = A_1 \sin(2\omega t + \delta) \text{ and } y = A_2 \sin \omega t$$

Where A_1 & A_2 are the amplitude of the x-vibrations & y-vibration and δ is the phase difference between them.

The equation of the curve of resultant motion is obtained as

$$\frac{x}{A_1} = \sin 2\omega t \cos\delta + \cos 2\omega t \sin\delta = 2\sin\omega t \cos\omega t \cos\delta + (1 - 2\sin^2 \omega t)\sin\delta$$

$$= 2\sin\omega t \sqrt{1 - \sin^2 \omega t}\cos\delta + (1 - 2\sin^2 \omega t)\sin\delta$$

$$\Rightarrow \frac{x}{A_1} = \frac{2y}{A_2}\sqrt{1 - \frac{y^2}{A_2^2}}\cos\delta + \left(1 - \frac{2y^2}{A_2^2}\right)\sin\delta \qquad \because \sin\omega t = \frac{y}{A_2}$$

$$\Rightarrow \frac{x}{A_1} - \left(1 - \frac{2y^2}{A_2^2}\right)\sin\delta = \frac{2y}{A_2}\sqrt{1 - \frac{y^2}{A_2^2}}\cos\delta$$

$$\Rightarrow \left(\frac{x}{A_1} - \sin\delta\right) + \frac{2y^2}{A_2^2}\sin\delta = \frac{2y}{A_2}\sqrt{1 - \frac{y^2}{A_2^2}}\cos\delta$$

Squaring and re-arranging terms, we get

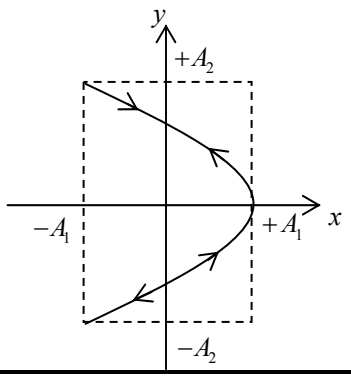
$$\left(\frac{x}{A_1} - \sin\delta\right)^2 + \frac{4y^2}{A_2^2}\left(\frac{y^2}{A_2^2} - 1 + \frac{x}{A_1}\sin\delta\right) = 0$$

This is the general equation of curve having two loops

(i) When $\delta = 0$, $\sin\delta = 0$, thus

$$\frac{x^2}{A_1^2} + \frac{4y^2}{A_2^2}\left(\frac{y^2}{A_2^2} - 1\right) = 0$$

This represents a curve symmetrical about both the axes.

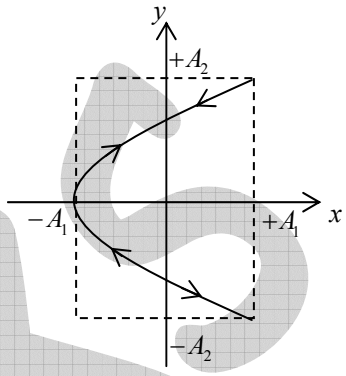


(ii) When $\delta = \frac{\pi}{2}$, $\sin\delta = 1$

$$\therefore \left(\frac{x}{A_1} - 1\right)^2 + \frac{4y^2}{A_2^2}\left(\frac{y^2}{A_2^2} - 1 + \frac{x}{A_1}\right) = 0 \text{ or } \left[\left(\frac{x}{A_1} - 1\right) + \frac{2y^2}{A_2^2}\right]^2 = 0$$

This represents two coincident parabolas symmetrical about the x-axis and their vertices at $(A_1, 0)$. The equation of Parabola is

$$y^2 = -\frac{A_2^2}{A_1^2}(x - A_1)$$



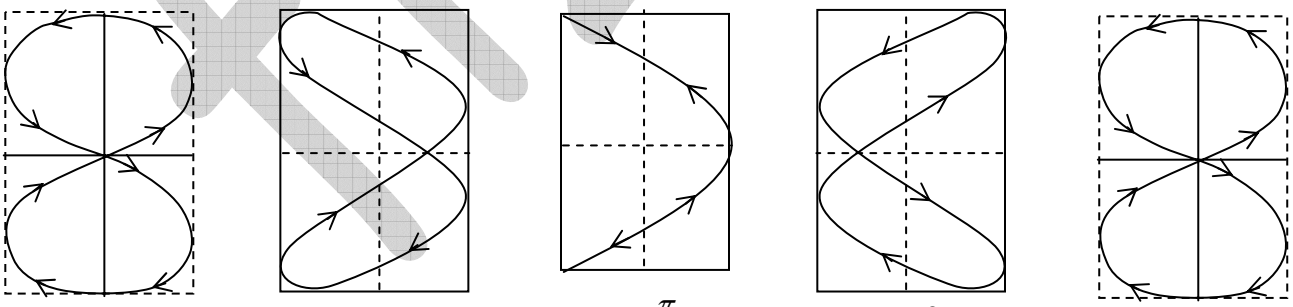
(iii) when $\delta = \pi$, $\sin\delta = 0$. The case is similar to case (i).

(iv) when $\delta = \frac{3\pi}{2}$, $\sin\delta = -1$

$$\therefore \left[\left(\frac{x}{A_1} + 1\right) - \frac{2y^2}{A_2^2}\right]^2 = 0 \quad \text{i.e. } y^2 = \frac{A_2^2}{2A_1}(x + A_1)$$

This represent parabola symmetric about the x-axis with their vertices at $(-A_1, 0)$.

Following figures shows the Lissajous figures for various initial phase differences for frequency ratio 2:1

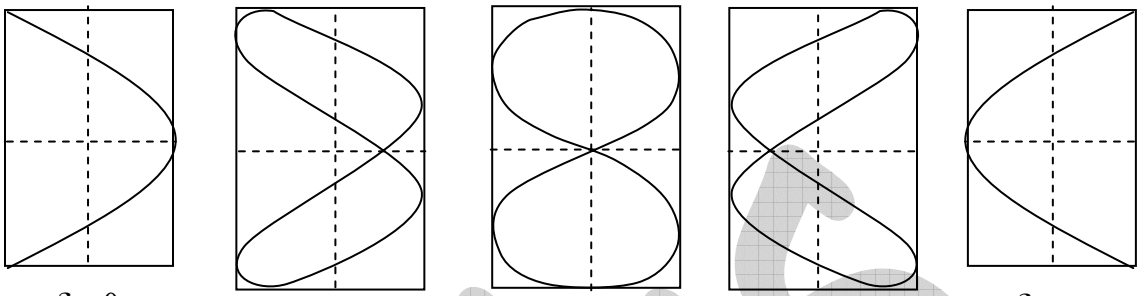


$\delta = 0$ $\quad$ $\delta = \frac{\pi}{4}$ $\quad$ $\delta = \frac{\pi}{2}$ $\quad$ $\delta = \frac{3\pi}{4}$ $\quad$ $\delta = \pi$

Note: If particles displacements are of following form

$$x = A_1 \sin(2\omega t) \quad \text{and} \quad y = A_2 \sin(\omega t + \delta)$$

The resultant Lissajous figures loops will be

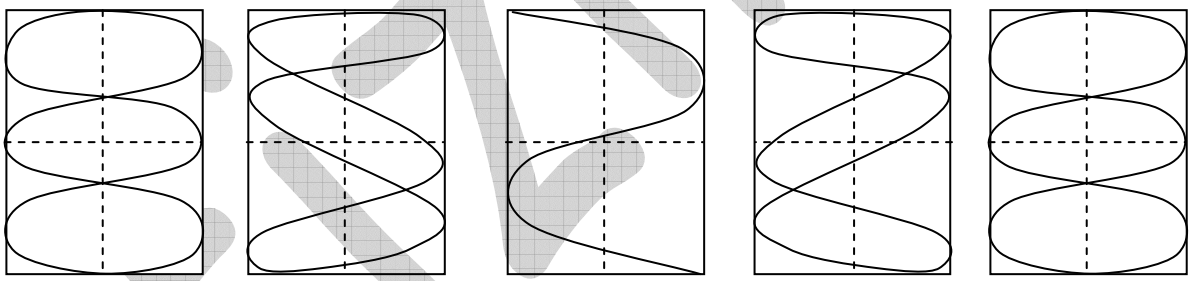


$\delta = 0$ $\quad \delta = \frac{\pi}{4}$ $\quad \delta = \frac{\pi}{2}$ $\quad \delta = \frac{3\pi}{4}$ $\quad \delta = \pi$

(2) Frequencies in the ratio of 3:1

For $x = A_1 \sin(3\omega t + \delta)$ and $y = A_2 \sin \omega t$

The resultant Lissajous figures at various initial phase differences are



$\delta = 0$ $\quad \delta = \frac{\pi}{4}$ $\quad \delta = \frac{\pi}{2}$ $\quad \delta = \frac{3\pi}{4}$ $\quad \delta = \pi$

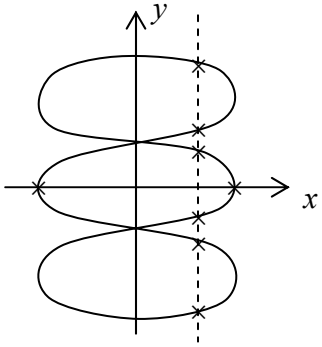
Method to find the frequencies ratio:

To find the frequencies ratio from the given Lissajous figure, draw two lines parallel to x and y axis which having maximum intercept with loops.

$$\frac{\text{Number of times the figure cuts the } y\text{-axis}}{\text{Number of times the figure cuts the } x\text{-axis}} = \frac{\text{Frequency of } x\text{-vibration}}{\text{Frequency of } y\text{-vibration}}$$

$$\frac{p_y}{p_x} = \frac{\omega_1}{\omega_2}$$

In above figure, $p_y = 6$ & $p_x = 2$ $\quad \Rightarrow \omega_1 : \omega_2 = 3:1$



MCQ (Multiple Choice Questions)

Q1. When two waves $y_1 = A\sin\left(\frac{2\pi x}{\lambda} - \omega t\right)$ and $y_2 = B\sin\left(\frac{2\pi x}{\lambda} - \omega t\right)$ interfere, the amplitude of the resulting wave is:

(a) $|A+B|$ (b) $|A^2 - B^2|^{1/2}$

(c) $|A^2 + B^2|^{1/2}$ (d) $\left[\frac{|A+B|^2 + |A-B|^2}{\sqrt{A^2+B^2}}\right]$

Q2. The amplitude of the vibrating particle due to superposition of two SHMs $y_1 = \sin\left(\omega t + \frac{\pi}{3}\right)$ and $y_2 = \sin\omega t$ is:

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

Q3. Two traveling waves $y_1 = 0.65\sin(0.4x - 800\,t)\ m$ and $y_2 = -0.65\sin(0.4x - 800t)\ m$ are superposed in a medium. For the resultant wave at the point $x = 2.5\pi\ m$, the maximum displacement is:

(a) zero (b) 1.30 m (c) 0.65 m (d) 0.910 m

Q4. A particle is subjected to two simple harmonic motions

$$x_1 = A_1\sin\omega t \quad \text{and} \quad x_2 = A_2\sin\left(\omega t + \frac{\pi}{4}\right)$$

The resultant simple harmonic motion will have amplitude of:

(a) $\frac{A_1 + A_2}{2}$ (b) $\sqrt{A_1^2 + A_2^2}$ (c) $\sqrt{A_1^2 + A_2^2 + \sqrt{2}A_1A_2}$ (d) $\sqrt{A_1^2 + A_2^2 + A_1A_2}$

Q5. A particle of mass m describes an elliptical orbit. This motion can be shown to be the sum of two simple harmonic motions at right angles to each other having:

(a) the same frequency but different amplitudes

(b) the same frequency and the same amplitude

(c) different frequencies but different amplitudes

(d) different frequencies and different amplitudes

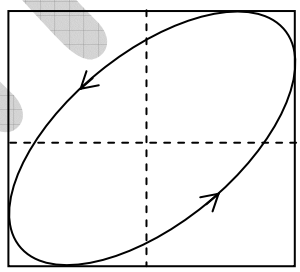
Q6. Two harmonic vibrations $x = a\sin(\omega t + \phi)$ and $y = b\sin\omega t$ represent the displacement of a particle along x and y axis, respectively. If $\phi = \frac{\pi}{2}$ then the trajectory of the particle will be:

(a) ellipse
(b) straight line with positive slope
(c) straight line with negative slope
(d) circle

Q7. Two simple harmonic waves of the same amplitudes and frequency differ by a phase $\frac{\pi}{2}$. When they are fed simultaneously to the X and Y plates of a CRO, the screen would display the trace of:

(a) a circle
(b) an ellipse
(c) a straight line
(d) a square

Q8. Lissjous figure shown above corresponds to which one of the following?



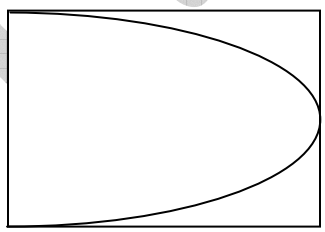
(a) Phase difference $\frac{\pi}{4}$ and period $1:1$

(b) Phase difference $\frac{\pi}{2}$ and period $1:1$

(c) Phase difference $\frac{3\pi}{4}$ and period $1:1$

(d) Phase difference π and period $1:1$

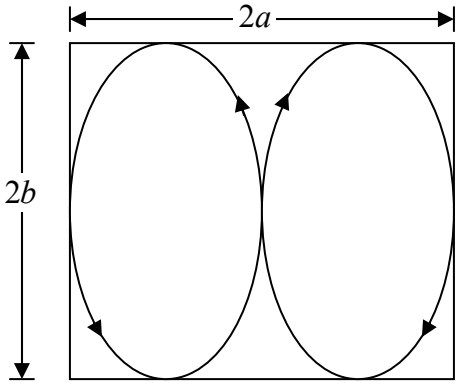
Q9.



Lissjous figure shown above corresponds to which one of the following?

(a) Phase difference $\frac{\pi}{2}$ and period $1:2$
(b) Phase difference $\frac{\pi}{4}$ and period $1:2$
(c) Phase difference $\frac{\pi}{2}$ and period $2:1$
(d) Phase difference $\frac{\pi}{4}$ and period $2:1$

Q10.



A particle subjected to two simple harmonic motions along x -axis and y -axis respectively describes the Lissajous figures as shown in the diagram. If ω_x and ω_y be the angular frequencies of these vibrations and δ be the phase difference between them, then:

(a) $\frac{\omega_x}{\omega_y}=2$ and $\delta=0$

(b) $\frac{\omega_x}{\omega_y}=2$ and $\delta=\frac{\pi}{2}$

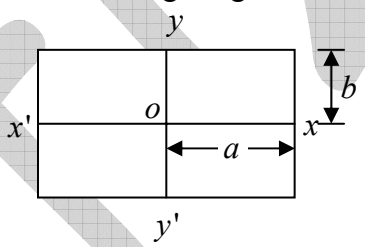
(c) $\frac{\omega_y}{\omega_x}=2$ and $\delta=0$

(d) $\frac{\omega_y}{\omega_x}=2$ and $\delta=\frac{\pi}{2}$

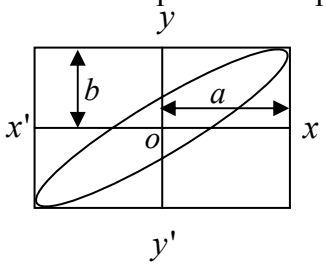
Q11. A particle is acted upon by two simple harmonic motions,

$$x=a\sin(\omega t+\phi) \quad \text{and} \quad y=b\sin\omega t$$

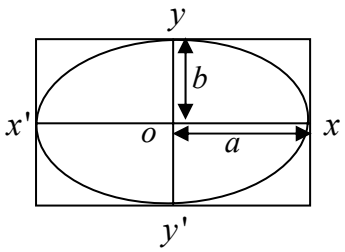
Which one of the following diagrams correctly gives the resultant path of the particle when $\phi=\frac{\pi}{2}$?



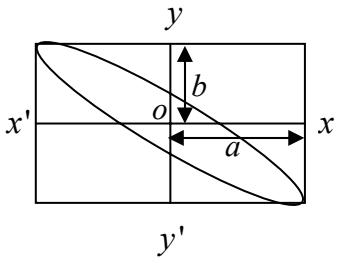
(a)



(b)



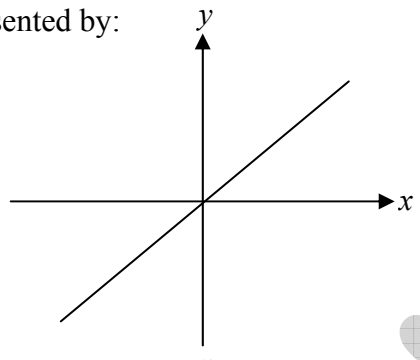
(c)



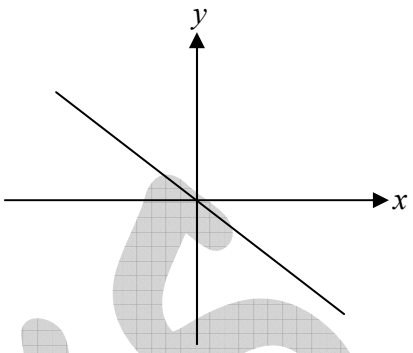
(d)

Q12. A particle is moving under the action of two simple harmonic forces. The forces are of equal magnitude and phase difference is $\left(\frac{\pi}{2}\right)$. The path taken by the particle would be represented by:

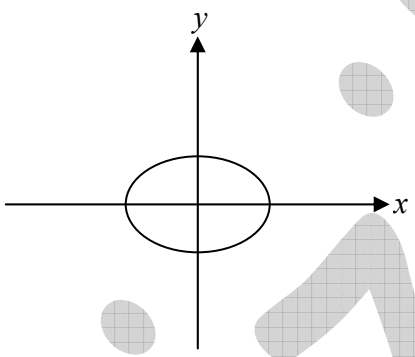
(a)



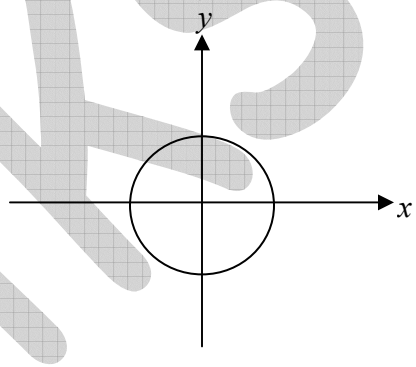
(b)



(c)



(d)



fiziks

MSQ (Multiple Select Questions)

Q13. Two traveling sinusoidal waves are described by the wave functions

$$y_1 = (5\ m)\sin[\pi(4.0x - 12t)] \quad \text{and} \quad y_2 = (5\,m)\sin[\pi(4.0x - 12t - 0.25)]$$

where x, y_1 & y_2 are in meters and t is in seconds. Which of the following statements are correct

(a) The resultant amplitude is $9.24\ m$

(b) The resultant amplitude is $10\ m$

(c) The frequency of the resultant wave is $6\,Hz$

(d) The frequency of the resultant wave is $4\ Hz$

Q14. A particle is simultaneously subjected to three SHM, all the same frequency and in the same direction. If the amplitude are $0.5\ mm$, $0.4\ mm$ and $0.3\ mm$ respectively and the phase difference between 1st and 2nd is 45° and between the second and the third is 30°, then

(a) The amplitude of resultant between 1st and 2nd motion is $0.5\ mm$

(b) The phase difference of the resultant of 1st and 2nd with 1st motion is 20°

(c) The amplitude of the resultant displacement is $1.03\ mm$

(d) The phase of the resultant motion relative to first motion is 33.7^o

Q15. Three simple harmonic motions in the same direction having the same amplitude a and same period are superposed. If each differs in phase from the next by 45^0, then

(a) the resultant amplitude is $(1+\sqrt{2})a$

(b) the phase of the resultant motion relative to the first is 90^0

(c) the energy associated with the resulting motion is $(3+2\sqrt{2})$ times the energy associated with any single motion

(d) the resulting motion is not simple harmonic

Q16. Two harmonic oscillations at right angles to each other with phase difference δ

$$x = A_1 \cos \omega t \quad \text{and} \qquad y = A_2 \cos(\omega t + \delta)$$

Superimpose each other. Which of the following statements are correct for the resultant motion of the particle?

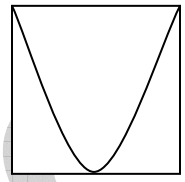
(a) at $\delta = \frac{3\pi}{2}$, the resulting motion is left handed elliptically linearly polarized vibration

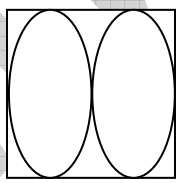
(b) at $\delta = 2\pi$, the resulting motion is linearly polarized vibration

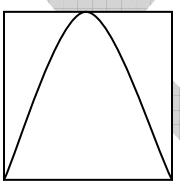
(c) at $\delta = \frac{\pi}{2}$, and $A_1 = A_2$, the resulting motion is left handed circularly polarized vibration

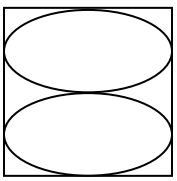
(d) at $\delta = \frac{3\pi}{4}$ and $A_1 = A_2$, the resulting motion is linearly polarized vibration

Q17. Two perpendicular SHM $x = A_1 \cos \omega t$ and $y = A_2 \cos(2\omega t + \delta)$ superimpose. Which of the followings are correct Lissajous figures at different phase difference?

(a) 
$\delta = 0$

(b) 
$\delta = \frac{\pi}{2}$

(c) 
$\delta = 0$

(d) 
$\delta = \frac{\pi}{2}$

NAT (Numerical Answer Type)

Q18. Two waves in one string are described by the wave functions

$$y_1 = 3\cos(4x - 2.6t) \qquad \text{and} \qquad y_2 = 4\sin(5x - 2t)$$

where y & x are in centimeter and t is in seconds. The resultant displacement of $y = y_1 + y_2$ at the point $x = 0.5$ and $t = 0$ is $\ldots\ldots cm$

Q19. A certain Lissajous pattern is produced by applying sinusoidal voltages to the vertical and horizontal input terminal of a CRO. The pattern makes five tangencies with the vertical and three with the horizontal. If the frequency of the input voltage is 12 kHz then the frequency of the signal applied to the vertical amplifier is $\ldots\ldots kHz$

Q20. The Lissajous figure obtained due to superposition of two mutually perpendicular vibrations of time period 1:1 is an oblique ellipse. Then the phase difference between them is …. ……in degree.

Q21. Two tuning forks, A and B, give beats/s when sound together. The frequency of A is 320 Hz. When some wax is added to B and sounded with fork A, 4 beats/s are heard again. The frequency of B (in Hz) is .....

Solution- MCQ (Multiple Choice Questions)

Ans. 1: (a)

Solution: we have by superposition theory

$$y = y_1 + y_2 = A\sin\left(\frac{2\pi x}{\lambda} - \omega t\right) + B\sin\left(\frac{2\pi x}{\lambda} - \omega t\right) = (A+B)\sin\left(\frac{2\pi x}{\lambda} - \omega t\right)$$

Hence, amplitude $R = (A+B)$

Ans. 2: (c)

Solution: By superposition principle

$$y = y_1 + y_2 = \sin\left(\omega t + \frac{\pi}{3}\right) + \sin\omega t \quad = 2\sin\left(\omega t + \frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$$

$$\Rightarrow y = 2\cos\left(\frac{\pi}{6}\right)\sin\left(\omega t + \frac{\pi}{6}\right) \Rightarrow \text{Amplitude } R = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$$

Ans. 3: (b)

Solution: The resultant $y = y_1 + y_2 \ = 0.65\sin(0.4x - 800t)\,m - 0.65\sin(0.4x - 800t)\,m$

$$\Rightarrow y = 0.65 \times 2\cos(0.4x)\sin 800t$$

$\Rightarrow$ The maximum displacement at any time $= 2 \times 0.65\cos(0.4x)$

when $x = 2.5\ \pi$, $y = 1.3\cos(2.5\pi \times 0.4) = 1.3\cos\pi = 1.3\ m$

Ans. 4: (c) Solution: Here $x_1 = A_1\sin\omega t$ and $x_2 = A_2\sin\left(\omega t + \frac{\pi}{4}\right)$

$$\Rightarrow x_2 = A_2\left[\sin\omega t\cos\frac{\pi}{4} + \cos\omega t\sin\frac{\pi}{4}\right] = A_2\left[\frac{\sin\omega t + \cos\omega t}{\sqrt{2}}\right]$$

Thus $x = x_1 + x_2 \ = A_1\sin\omega t + \frac{A_2}{\sqrt{2}}(\sin\omega t + \cos\omega t) \Rightarrow x = \left(A_1 + \frac{A_2}{\sqrt{2}}\right)\sin\omega t + \frac{A_2}{\sqrt{2}}\cos\omega t$

Let $R\cos\phi = A_1 + \frac{A_2}{\sqrt{2}}$ and $R\sin\phi = \frac{A_2}{\sqrt{2}}$ $\Rightarrow x = R\sin(\omega t + \phi)$

$$\Rightarrow R = \sqrt{A_1^2 + A_2^2 + \sqrt{2}A_1A_2}$$

Ans. 5: (a) Solution: If two SHMs of same frequencies but different amplitudes oscillate perpendicular to each other then resultant is elliptical SHM.

Ans. 6: (a) Solution: The resultant of two perpendiculars SHM given as

$x = a\sin(\omega t + \phi); y = b\sin\omega t$ is a curve known as Lissajous figure.

If $\phi = \frac{\pi}{2}$ then $x = a\sin\left(\omega t + \frac{\pi}{2}\right) \Rightarrow x = a\cos\omega t$, $y = b\sin\omega t$

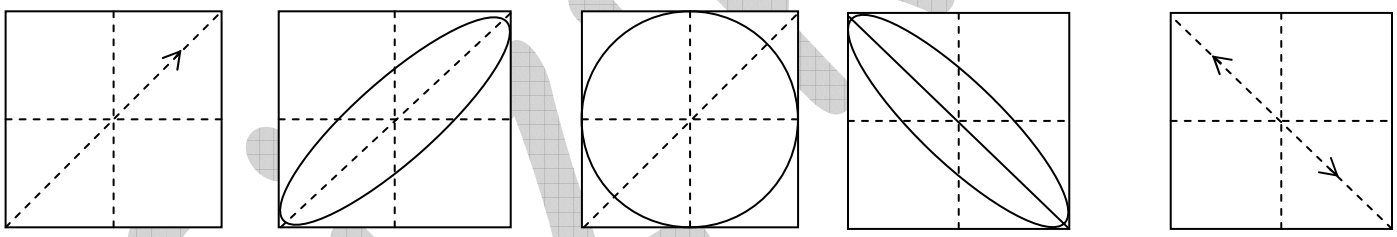
Squaring and adding, we get $\cos^2\phi + \sin^2\phi = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

This is equation of an ellipse.

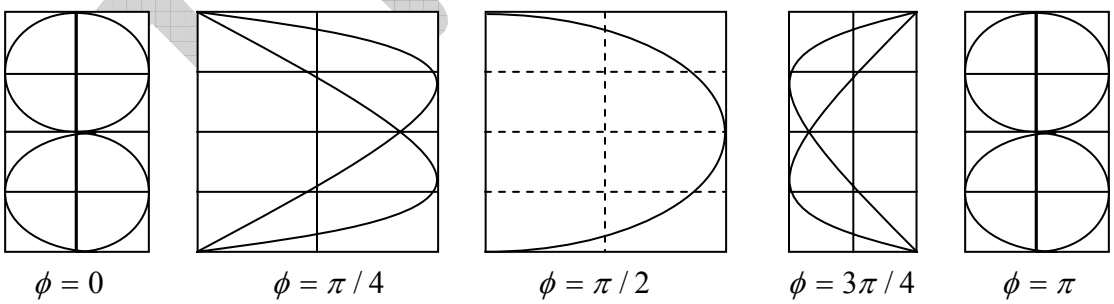
Ans. 7: (a)

Ans. 8: (a)

Solution:



Ans. 9: (c) $\frac{\omega_x}{\omega_y} = \frac{2}{1}$



Ans. 10: (c) Solution: $\frac{\omega_x}{\omega_y} = \frac{2}{4} \Rightarrow \frac{\omega_y}{\omega_x} = 2$ and $\delta = 0$

Ans. 11: (c) Solution: we have $x = a\sin(\omega t + \phi) \Rightarrow x = a\cos\omega t \quad \because \phi = 90^0$ and $y = b\sin\omega t$

squaring and adding $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Ans. 12: (d)

MSQ (Multiple Select Questions)

Ans. 13: (a) and (c)

Solution: Let $y_1 = A\sin(kx - \omega t)$ and $y_2 = A\sin(kx - \omega t - \phi)$

$$y = y_1 + y_2 = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

Where resultant amplitude is $R = 2A\cos\frac{\phi}{2} = 2\times 5\cos\left[\frac{\frac{\pi}{4}}{2}\right] = 9.24$

and $f = \frac{\omega}{2\pi} = \frac{12\pi}{2\pi} = 6\ Hz$.

The correct options are (a) & (c) only

Ans. 14: (b), (c) and (d)

Solution: Here $A_1 = 0.5\ mm,\ A_2 = 0.4\ mm$, phase difference between them is $\phi = \phi_2 - \phi_1 = 45^o$

The resultant amplitude A is

$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\phi = (0.5)^2 + (0.4)^2 + 2\times 0.5\times 0.4\times\cos 45^o \Rightarrow A = 0.83\ mm$

Thus option (a) is not correct

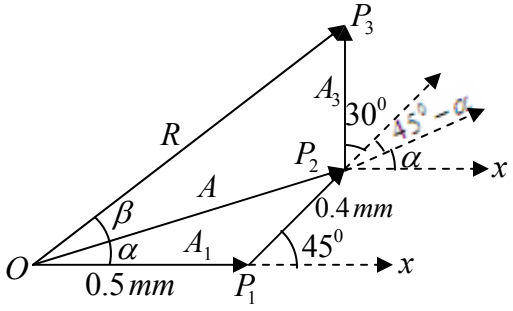
Phase difference of the resultant OP_1 and P_1P_2 relative to the 1st component OP_1 is α

In ΔOP_1P_2 we have

$$\frac{\sin\alpha}{P_1P_2} = \frac{\sin\angle OP_1P_2}{A}$$

$$\therefore \quad \sin\alpha = \sin 45^o \times \frac{P_1P_2}{A} = \sin 45^o \times \frac{0.4}{0.8324} = 0.3398$$

$$\Rightarrow \alpha = 19.87^o$$



Thus option (b) is correct

Now the phase difference between OP_2 and $P_2P_3 = \phi'$

$$\Rightarrow \phi' = 30^o + \left(45^o - 19.87^o\right) = 55.13^o$$

$$\therefore \quad R^2 = A^2 + A_3^2 + 2AA_3\cos\phi' = \left(0.8324\right)^2 + \left(0.3\right)^2 + 2\times 0.8324\times 0.3\times\cos\left(55.13^o\right)$$

$$R^2 = 1.068\ mm^2 \Rightarrow R = 1.03\ mm$$

Thus option (c) is correct.

and in OP_2P_3; $\quad \dfrac{\sin\beta}{P_2P_3} = \dfrac{\sin\angle OP_2P_3}{OP_3}$

$$\Rightarrow \sin\beta = \sin\left(180^o - 55.13^o\right)\times\frac{P_2P_3}{OP_3} = \sin\left(124.87^o\right)\times\frac{0.3}{1.03} = 0.239 \Rightarrow \beta = 13.8^o$$

Hence $\delta = \alpha + \beta = 33.7^o$

Thus option (d) is correct.

Ans. 15: (a) and (c)

Solution: The resultant of the given three simple harmonic motions is

$$y = y_1 + y_2 + y_3 = a\sin\omega t + a\sin\left(\omega t + 45^0\right) + a\sin\left(\omega t + 90^0\right)$$

$$\Rightarrow y = a\sin\omega t + a\left(\sin\omega t\cos 45^0 + \cos\omega t\sin 45^0\right) + a\cos\omega t$$

$$\Rightarrow y = a\left(\sqrt{2}+1\right)\left(\frac{\sin\omega t + \cos\omega t}{\sqrt{2}}\right) = a\left(\sqrt{2}+1\right)\sin\left(\omega t + 45^o\right)$$

$$\Rightarrow y = A\sin\left(\omega t + 45^0\right)$$

The resultant is also a simple harmonic motion with amplitude A equals to $a\left(\sqrt{2}+1\right)$

The energy associated with the resultant motion is $\frac{1}{2}mA^2\omega^2$

The energy associated with any single motion is $\frac{1}{2}ma^2\omega^2$

Hence, $\quad \dfrac{E_{\text{resultant}}}{E_{\text{single}}} = \dfrac{A^2}{a^2} = \dfrac{a^2\left(\sqrt{2}+1\right)^2}{a^2} = \left(3+2\sqrt{2}\right)$

Ans. 16: (a) and (b)

Solution: For $x = A_1 \cos \omega t$ and $y = A_2 \cos(\omega t + \delta)$

The resulting motion is described by $\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} \cos\delta = \sin^2 \delta$

(a) at $\delta = \frac{3\pi}{2}$, $\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$

which is ellipse but the motion is counter clockwise. Correct option

(b) at $\delta = 2\pi$, $\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} = 0 \Rightarrow \left(y - \frac{A_2}{A_1} x\right) = 0 \Rightarrow y = + \frac{A_2}{A_1} x$

This is equation of line. Correct option

(c) at $\delta = \frac{\pi}{2}$, and $A_1 = A_2 \quad \Rightarrow x^2 + y^2 = A^2$

This is equation of circle but motion is clockwise. Wrong option

(d) at $\delta = \frac{3\pi}{4}$, The resulting equation represent general ellipse and not linearly polarized.

Option is incorrect

Thus correct options are only (a) & (b)

Ans. 17: (a) and (b)

Solution: Resultant of $x = A_1 \cos \omega t$ and $y = A_2 \cos(2\omega t + \delta)$ gives

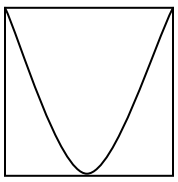
$$\left(\frac{y}{A_2} + \cos\delta\right)^2 + \frac{4x^2}{A_1^2}\left(\frac{x^3}{A_1^2} - 1 - \frac{y}{A_2}\cos\delta\right) = 0$$

at $\delta = 0$; If figure $x^2 = \frac{A_1^2}{2A_2}(y + A_2)$. The resultant motion of the particle is

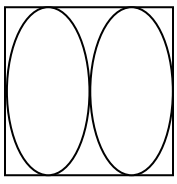
whereas at $\delta = \frac{\pi}{2}$, If becomes

$$\frac{y^2}{A_2^2} + \frac{4x^2}{A_1^2}\left(\frac{x^2}{A_1^2} - 1\right) = 0$$

which is represented as



$\delta = 0$



$\delta = \frac{\pi}{2}$

NAT (Numerical Answer Type)

Ans. 18: 1.15

Solution: $y = y_1 + y_2 = 3\cos(4x - 1.6t) + 4\sin(5x - 2t)$ at $x = 0.5$ & $t = 0$

$$y = 3\cos(+2\,rad) + 4.0\sin(2.5\ rad) = 3\times -0.416 + 4\times 0.599 = 1.15\ cm$$

Ans. 19: 7.2

Solution: $\dfrac{\text{Number of points at which figure is tangent to vertical line}}{\text{Number of points at which figure is tangent horizontal line}} = \dfrac{\omega_x}{\omega_y}$

$$\Rightarrow \quad \frac{5}{3} = \frac{12\ kHz}{\omega_y} \quad \Rightarrow \quad \omega_y = \frac{36}{5} = 7.2\ kH_2$$

Ans. 20: 45

Solution: The resultant of two waves $x = A_1\sin(\omega t + \delta)$ and $y = A_2\sin\omega t$

is $x + y = \dfrac{x^2}{A_1^2} + \dfrac{y^2}{A_2^2} + \dfrac{2xy}{A_1A_2}\cos\delta + \sin^2\delta$

at $\delta = \dfrac{\pi}{4}$ above equation as becomes $\dfrac{x^2}{A_1^2} + \dfrac{y^2}{A_2^2} + \dfrac{2xy}{A_1A_2}\times\dfrac{1}{\sqrt{2}} = \dfrac{1}{2}$

which is the equation of oblique ellipse

Ans. 21: 324

Solution: The frequency of $B = 320 \pm 4 = 316$ or 324.

When some wax is added to B its frequency becomes lower and since again beat produced with A is 4 so, frequency of B cannot be 316 and hence, frequency of B (in Hz) is 324.

4. Wave Motion

A wave is defined as a oscillatory disturbance of particle in the medium which distribute energy from one particle to other particle.

In general there are only two types of waves one which required material medium to travel while other travel in vacuum also. The former is called mechanical wave and later is called electromagnetic wave.

Mechanical Wave

Water wave, sound wave, ultrasound wave, earthquake wave etc

Electromagnetic wave

Radio wave, Microwave, Infrared wave, light wave etc.

There are two distinct classes of wave

(i) Transverse wave and (ii) Longitudinal wave

Transverse wave

In the transverse wave motion the oscillation of the particles is at right angle to the direction of propagation of the wave.

Example: Wave generated in rope and stretched string, electromagnetic wave, water wave, earthquake wave etc.

Longitudinal Wave

In the longitudinal wave the oscillation of the particle is the same as the direction of propagation of the wave.

Example: Sound wave, ultrasound wave etc.

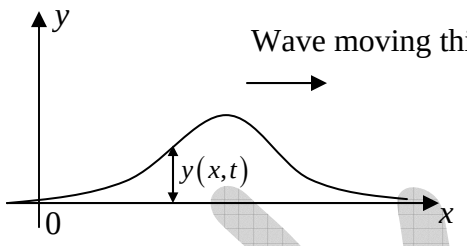
Wave Front

A plane or a surface on which all the particles of the medium are in an identical state of motion at a given instant, i.e. in same phase, is called a wave front.

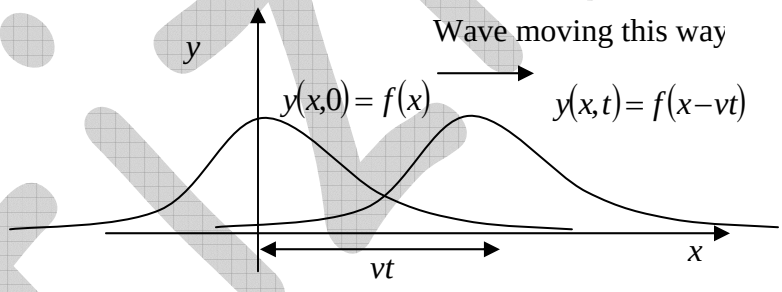
In homogeneous and isotropic medium, the wave front is always perpendicular to the direction of the propagation of wave. Wave front although have various shape but the two most important shapes are Plane Wave Front and Spherical Wave Front.

4.1 Traveling Waves

Let's start with a rope, like a clothesline, stretched between two hooks. You take one end off the hook, holding the rope, and, keeping it stretched fairly tight, wave your hand up and back once. If you do it fast enough, you'll see a single bump travel along the rope: This is the simplest example of a ***traveling wave***.



Taking the rope to be stretched tightly enough that we can take it to be horizontal, we'll use it as x-axis. The y-axis is taken vertically upwards, and we only wave the rope in an up-and-down way, so actually $y(x,t)$ will be how far the rope is from its rest position at x at time t : that is, the graph $y(x,t)$ above just shows where the rope is at time t.



Taking for convenience time $t = 0$ to be the moment when the peak of the wave passes $x = 0$, we graph here the rope's position at t = 0, and some later time t. Denoting the first function by $y(x,0) = f(x)$, than the displacement y of the rope at any horizontal position at x at time t has the form

$$y(x,t) = f(x - vt)$$

it's *the same function*—the "same shape"—but moved over by vt, where v is the velocity of the wave in the positive x axis.

Expression for a Plane Progressive Harmonic Wave

Wave equation can be expressed in following types $y = a\sin\omega\left(t - \frac{x}{v}\right)$

In term of time period $T = \frac{2\pi}{\omega}$ and wavelength $\lambda = vT$ is $y = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$

In term of wave vector $k = \frac{2\pi}{\lambda}$ is

$y = a\sin\frac{2\pi}{\lambda}(vt - x)$ or $y = a\sin k(vt - x)$ or $y = a\sin(\omega t - kx)$

Differential Equation of Wave Motion

We have the equation of plane progressive wave motion is

$$y = a\sin\frac{2\pi}{\lambda}(vt - x)$$

acceleration of the particle $= \frac{\partial^2 y}{\partial t^2} = -\frac{4\pi^2 v^2}{\lambda^2} a\sin\frac{2\pi}{\lambda}(vt - x) = -\frac{4\pi^2 v^2}{\lambda^2} y$

Also, rate of change of compression with distance can be obtain as

$$\frac{\partial^2 y}{\partial x^2} = -\frac{4\pi^2}{\lambda^2} a\sin\frac{2\pi}{\lambda}(vt - x) = -\frac{4\pi^2}{\lambda^2} y$$

Combining above two equation, we get

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

This is the differential equation for one dimensional wave motion. We can also write

partilce acceleration = $(\text{wave velocity})^2 \times$ curvature of the dispacement curve

Energy Density of the Plane progressive Wave

In a progressive wave motion, energy from the source is passed from particles to other particles. There is a regular transmission of energy across every section of the medium.

Energy density of plane progressive wave = total energy

= {kinetic energy (K.E) + potential energy (P.E.)} per unit volume of the medium.

Let ρ is the density of the medium in which wave is moving.

Thus the kinetic energy per unit volume K_ρ is obtained as

$$K.E. = \frac{1}{2}(\text{mass})(\text{velocity})^2 = \frac{1}{2}(\text{density}\times\text{velocity})(\text{velocity})$$

$$K_\rho = \frac{K.E.}{\text{volume}} = \frac{1}{2}(\text{density})(\text{velocity})^2 = \frac{2\pi^2 v^2\rho}{\lambda^2}a^2\cos^2\frac{2\pi}{\lambda}(vt-x)$$

The potential energy per unit volume U_ρ is obtained as

$$\text{Force acting perunit volume} = \text{density}\times\text{acceleration} = \rho\frac{\partial^2 y}{\partial t^2} = \frac{4\pi^2 v^2\rho}{\lambda^2}y$$

Now, work done per unit volume in a small displacement dy

$$= \text{Force per unit volume}\times\text{displacement} = \frac{4\pi^2 v^2\rho}{\lambda^2}ydy$$

Work done per unit volume for whole displacement from 0 to y

$$= \int_0^y \frac{4\pi^2 v^2\rho}{\lambda^2}ydy = \frac{2\pi^2 v^2\rho}{\lambda^2}y^2 = \frac{2\pi^2 v^2\rho}{\lambda^2}a^2\sin^2\frac{2\pi}{\lambda}(vt-x)$$

This work done is stored in the medium in the form of potential energy.

Thus $U_\rho = P.E.\text{per unit volume} = \frac{2\pi^2 v^2\rho}{\lambda^2}a^2\sin^2\frac{2\pi}{\lambda}(vt-x)$

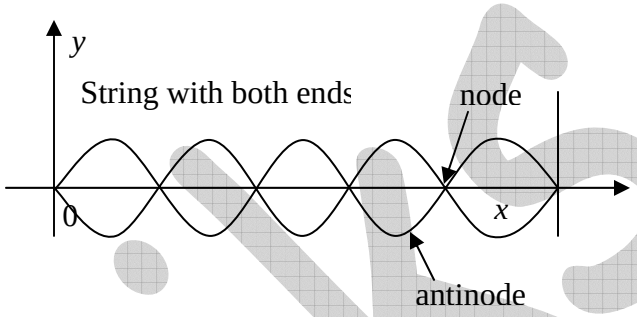
The total energy per unit volume of the medium or the energy density of the plane progressive wave is $\mathrm{E}_\rho = K_\rho + U_\rho$

$$E_\rho = \frac{2\pi^2 v^2\rho}{\lambda^2}a^2\cos^2\frac{2\pi}{\lambda}(vt-x) + \frac{2\pi^2 v^2\rho}{\lambda^2}a^2\sin^2\frac{2\pi}{\lambda}(vt-x)$$

$$E_\rho = \frac{2\pi^2 v^2\rho}{\lambda^2}a^2 = 2\pi^2\left(\frac{v}{\lambda}\right)\rho a^2 = 2\pi^2 f^2 a^2\rho$$ where, f is the frequency of the wave.

4.2 Stationary Waves

Another familiar kind of wave is that generated on a string fixed at both ends when it is made to vibrate. We found in class that for certain frequencies the string vibrated in a sine-wave pattern, as illustrated below, with no vibration at the ends, of course, but also no vibration at a series of equally-spaced points between the ends: these quiet places we term ***nodes***. The places of maximum oscillation are ***antinodes***.



Almost all musical instruments generate standing waves: the piano has standing waves on strings, the organ generates standing waves in the air in pipes. Other instruments are more complicated: although the sound of a violin comes from a vibrating string, resonance with the rest of the instrument gives rise to complicated standing wave patterns.

When two identical wave either longitudinal or transverse, travel through a medium along the same line in opposite direction, they superimpose to produce a wave which appears to be stationary in space. Such waves are called stationary wave or standing wave.

Condition for Stationary waves

The presence of bounded medium is prerequisite for producing stationary wave. Medium should not be infinite in length, it should have boundary.

If the medium is infinite, the wave just continues to travel along it for an infinite time, i.e. such medium support travelling wave.

General Consideration of Stationary Wave

Consider a plane progressive wave of amplitude a travelling with velocity v along the $+x$ axis. Its equation is written as $y_1 = a\sin\frac{2\pi}{\lambda}(vt - x)$

where, y_1 is the displacement at a point x at a time t and a is the amplitude.

If this wave is reflected from a rigid boundary wall situated at $x = 0$. The displacement y_2 of the reflected wave will be $y_2 = a'\sin\frac{2\pi}{\lambda}(vt + x)$

Where, a' is the amplitude of the reflected wave.

Both the wave travelling in the same linear path superimpose and the equation of the resultant wave is $y = y_1 + y_2 = a\sin\frac{2\pi}{\lambda}(vt - x) + a'\sin\frac{2\pi}{\lambda}(vt + x)$

The displacement y at rigid wall (at $x = 0$) will always be zero. This give the boundary condition, $y = 0$ at $x = 0$.

Thus $0 = a\sin\frac{2\pi}{\lambda}(vt) + a'\sin\frac{2\pi}{\lambda}(vt) \quad \Rightarrow \quad a = -a'$

Thus the resultant displacement is

$$y = a\left[\sin\frac{2\pi}{\lambda}(vt - x) - \sin\frac{2\pi}{\lambda}(vt + x)\right] = -2a\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda} \Rightarrow y = -2a\sin kx\cos\omega t$$

This is the equation of the resultant stationary wave. Equation shows that the resulting wave is also a simple harmonic wave of the same time period and wavelength but with the amplitude of $-2a\sin kx$.

The particle velocity is

$$u = \frac{\partial y}{\partial t} = \frac{4\pi av}{\lambda}\sin\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$$

The resultant pressure variation is

$$p = -K\frac{\partial y}{\partial x} = K\frac{4\pi a}{\lambda}\cos\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$$

Let us now consider the changes that occur,

(a) with respect to position of a particle, (b) with respect to time.

(a) Changes with respect to position

(i) At point where

$$\frac{2\pi x}{\lambda} = 0, \pi, 2\pi, 3\pi.... \quad \text{or} \qquad x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}....$$

we have $\sin\left(\frac{2\pi x}{\lambda}\right) = 0$ and $\cos\left(\frac{2\pi x}{\lambda}\right) = \pm 1$

At these values of x ; Displacement $(y) = 0$, velocity $(u) = 0$, while the pressure (p) is maximum (alternatively positive and negative) relative to all other points. These points are called **NODES** and are separated from each other by $\frac{\lambda}{2}$.

(ii) At point where

$$\frac{2\pi x}{\lambda} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}... \quad \text{or} \qquad x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}...$$

We have $\sin\left(\frac{2\pi x}{\lambda}\right) = \pm 1$ and $\cos\left(\frac{2\pi x}{\lambda}\right) = 0$

At these values of x; Displacement (y) and velocity (u) is always maximum, while the pressure variation (p) is always zero relative to all other points. These points are called **ANTINODES** and are separated from each other by $\frac{\lambda}{2}$.

(b) Changes with respect to time

(i) At a time when

$$\frac{2\pi vt}{\lambda} = \frac{2\pi t}{T} = 0, \pi, 2\pi, 3\pi.... \quad \text{or} \quad t = 0, \frac{T}{2}, \frac{2T}{2}, \frac{3T}{2}....$$

We have $\sin\left(\frac{2\pi vt}{\lambda}\right) = 0$ and $\cos\left(\frac{2\pi vt}{\lambda}\right) = \pm 1$

At these points, the displacement y and pressure variation p are maximum while velocity u is zero relative to all other times. Such instant occur twice in each period

(ii) At a time when

$$\frac{2\pi vt}{\lambda}=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2}\ldots \quad \text{or} \qquad t=\frac{T}{4},\frac{3T}{4},\frac{5T}{4}\ldots$$

We have $\sin\left(\frac{2\pi vt}{\lambda}\right)=\pm 1$ and $\cos\left(\frac{2\pi vt}{\lambda}\right)=0$

At these points, the displacement *y* and pressure variation *p* are zero while velocity *u* is maximum relative to all other times. Such instant occur twice in each period.

Note: Superposition of incident and reflected wave produce a stationary weaves characterized by nodes which remains permanently at rest but suffer maximum pressure variation, and the antinodes which have maximum displacement but suffer minimum pressure variation.

We now, are going to discuss the formation of stationary wave and excitation of different frequency modes in following different three cases.

(i) When both the boundaries are rigid

(ii) When one boundary is rigid and other is free

(iii) When both the boundaries are free

Case I: When both the boundaries of the medium are rigid

Let two wave travelling in the same linear path in opposite direction in medium of length *l*, superimpose and the equation of the resultant wave is

$$y=a\sin\frac{2\pi}{\lambda}(vt-x)+a'\sin\frac{2\pi}{\lambda}(vt+x)$$

First boundary condition: the displacement $y=0$ at $x=0$. This gives

$$0=a\sin\frac{2\pi}{\lambda}(vt)+a'\sin\frac{2\pi}{\lambda}(vt) \qquad \Rightarrow a=-a'$$

Thus the resultant displacement is $y=-2a\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$

This is the expression of standing wave

Second boundary condition: the displacement $y = 0$ at $x = l$. this gives

$$0 = -2a \sin \frac{2\pi l}{\lambda} \cos \frac{2\pi vt}{\lambda}$$

This must be true for all values of *t*, thus

$$\sin \frac{2\pi l}{\lambda} = 0 \Rightarrow \frac{2\pi l}{\lambda} = n\pi \Rightarrow \lambda = \frac{2l}{n} \quad \text{where } n = 0,1,2,3.....$$

The allowed frequencies in such medium is $f = \frac{v}{\lambda} = \frac{nv}{2l}$

This shows that the when medium has rigid boundaries at both end, it sustain vibration of certain discrete frequencies such as $f = \frac{v}{2l}, \frac{2v}{2l}, \frac{3v}{2l}, \frac{4v}{2l}, \frac{5v}{2l}.....$

These frequencies are the normal frequencies or modes of vibration. The lowest frequency is the fundamental frequency. Since frequencies are in harmonic series, they are also called Harmonics.

Conclusion: In a linear bounded medium when both the boundaries are rigid, all the harmonics even and odd are present.

Case II: When one boundary of the medium is rigid and other is free

Consider that boundary is rigid at $x = 0$, and the other boundary is free at $x = l$. The boundary conditions for such medium is $y = 0$ at $x = 0$ and pressure $p = -K\frac{\partial y}{\partial x} = 0$ at $x = l$.

The equation of the resultant wave is $y = a \sin \frac{2\pi}{\lambda}(vt - x) + a' \sin \frac{2\pi}{\lambda}(vt + x)$

First boundary condition: the displacement $y = 0$ at $x = 0$. This gives

$$0 = a \sin \frac{2\pi}{\lambda}(vt) + a' \sin \frac{2\pi}{\lambda}(vt) \Rightarrow a = -a'$$

Thus the resultant displacement is $y = -2a \sin \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda}$

The resultant pressure variation is $p = -K\frac{\partial y}{\partial x} = K\frac{4\pi a}{\lambda} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda}$

Second boundary condition: $p=-K\frac{\partial y}{\partial x}=0$ at $x=l$, we get

$$0=\frac{4\pi a}{\lambda}\cos\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$$

This must be true for all values of t, thus

$$\cos\frac{2\pi l}{\lambda}=0\Rightarrow\frac{2\pi l}{\lambda}=(2n+1)\frac{\pi}{2}\Rightarrow\lambda=\frac{4l}{2n+1}\qquad\text{where } n=0,1,2,3....$$

The allowed frequencies are $f=\frac{v}{\lambda}=\frac{v}{4l},\frac{3v}{4l},\frac{5v}{4l},\frac{7v}{4l}....$

The fundamental frequency is $\frac{v}{4l}$ and the higher frequencies are odd multiples of it.

Conclusion: In a linear bounded medium when one boundary is rigid and other free, only odd harmonics are present.

Case III: When both the boundaries are free

Consider a medium with free boundaries at $x=0$ and at $x=l$. The boundary conditions for such medium are pressure $p=-K\frac{\partial y}{\partial x}=0$ at $x=0$ and at $x=l$. The equation of the resultant wave is $y=a\sin\frac{2\pi}{\lambda}(vt-x)+a'\sin\frac{2\pi}{\lambda}(vt+x)$

The pressure variation is

$$p=-K\frac{\partial y}{\partial x}=-K\left[-\frac{2\pi a}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x)+\frac{2\pi a'}{\lambda}\cos\frac{2\pi}{\lambda}(vt+x)\right]$$

First boundary condition: $p=-K\frac{\partial y}{\partial x}=0$ at $x=0$, we get

$$0=-\frac{2\pi a}{\lambda}\cos\frac{2\pi}{\lambda}(vt)+\frac{2\pi a'}{\lambda}\cos\frac{2\pi}{\lambda}(vt)\Rightarrow\text{this gives}\,a=a'$$

Thus the resultant wave is

$$y=a\left[\sin\frac{2\pi}{\lambda}(vt-x)+\sin\frac{2\pi}{\lambda}(vt+x)\right]=2a\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$$

The resultant pressure variation is $p = -K\frac{\partial y}{\partial x} = K\frac{4\pi a}{\lambda}\sin\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$

Second boundary condition: $p = -K\frac{\partial y}{\partial x} = 0$ at $x = l$, we get

$$0 = \frac{4\pi a}{\lambda}\sin\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$$

This must be true for all values of t, thus

$$\sin\frac{2\pi l}{\lambda} = 0 \quad \Rightarrow \frac{2\pi l}{\lambda} = n\pi \Rightarrow \lambda = \frac{2l}{n} \qquad \text{where } n = 0,1,2,3....$$

The allowed frequencies in such medium is $f = \frac{v}{\lambda} = \frac{nv}{2l} = \frac{v}{2l}, \frac{2v}{2l}, \frac{3v}{2l}, \frac{4v}{2l}, \frac{5v}{2l}.....$

The fundamental frequency is $\frac{v}{2l}$, and the higher frequencies are integral multiple of it.

Conclusion: In a linear bounded medium when both boundaries are free, all the harmonics are present.

Example: The vibrations of a string of length 60 cm fixed at both ends are represented by the equation

$y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t)$ where x and y are in cm and t in second.

(i) What is the maximum displacement of a point at $x = 5\,cm$?

(ii) Where are the nodes located along the string?

(iii) What is the velocity of the particle at $x = 7.5\,cm$, $t = 0.25\sec$?

(iv) Write down the equations of the component waves whose superposition gives the above wave.

Solution:

(i) The equation represents a stationary wave having amplitude $A = 4\sin\left(\frac{\pi x}{15}\right)$

At $x = 5$, amplitude $= 4\sin\left(\frac{\pi}{3}\right) = 2\sqrt{3} = 3.46\,cm$ (i)

(ii) Nodes along string:

At nodes, amplitude $A=0\ \therefore \sin\left(\frac{\pi x}{15}\right)=0 \Rightarrow \frac{\pi x}{15}=n\pi$ where $n=0,1,2,....$

or $x=15n$ or $x=0,\ 15\,cm,\ 30\,cm,\ 60\,cm$.

(iii) Velocity of particle:

$$y=4\sin\left(\frac{\pi x}{15}\right)\cos\left(96\pi t\right)\ \ \therefore\frac{dy}{dt}=4\sin\left(\frac{\pi x}{15}\right)\left[-96\pi\times\sin\left(96\pi t\right)\right]$$

At $x=7.5\,cm, t=0.25\sec.$,

$$v=4\sin\left(\frac{\pi\times7.5}{15}\right)\left[-96\pi\times\sin\left(96\pi\times0.25\right)\right]$$

$$\therefore \text{Velocity}=4\sin\left(\frac{\pi}{2}\right)\left[-96\pi\times\sin\left(24\pi\right)\right]=0$$

(iv) Component waves:

$$y=4\sin\left(\frac{\pi x}{15}\right)\cos\left(96\pi t\right)\text{ or }\ y=2\times\left[2\cos\left(96\pi t\right)\sin\left(\frac{\pi x}{15}\right)\right]$$

$$\text{or }\ y=2\left[\sin\left(96\pi t+\frac{\pi x}{15}\right)-\sin\left(96\pi t-\frac{\pi x}{15}\right)\right]$$

$\therefore$ The two component waves are

$$y_1=2\sin\left(96\pi t+\frac{\pi x}{15}\right)\ \text{ and }\ y_2=-2\sin\left(96\pi t-\frac{\pi x}{15}\right).$$

Example: The following equations represent transverse waves:

$$z_1=A\cos\left(kx-\omega t\right); z_2=A\cos\left(kx+\omega t\right);\ z_3=A\cos\left(ky-\omega t\right)$$

Identify the combination of the waves which will produce

(i) standing wave, (ii) a wave travelling in the direction making an angle 45° with the positive x and positive y axes.

In each case, find the positions at which the resultant intensity is always zero.

Solution: $z_1 = A\cos(kx - \omega t)$ represents a wave travelling along positive x-axis.

$z_2 = A\cos(kx + \omega t)$ represents a wave travelling along negative x-axis.

$z_3 = A\cos(ky - \omega t)$ represents a wave travelling along positive y-axis.

(i) Standing waves

z_1 and z_2 denote waves travelling in opposite directions along x-axis. They superpose to produce standing waves.

$$z = z_1 + z_2 \quad \text{or} \quad z = 2A\cos kx\cos\omega t$$

Amplitude of stationary waves $= (2A\cos kx)$

For resultant intensity to be zero, $2A\cos kx = 0$ or $x = \dfrac{(2n+1)\pi}{2k}$ where $n = 0, 1, 2, ...$

(ii) A wave at 45° :

By superposition of z_1 and z_3, a wave will be formed which will travel in the direction making an angle of 45° with the positive x and positive y axes.

$\therefore z = z_1 + z_3$

or $z = A\cos(kx - \omega t) + A\cos(ky - \omega t)$

or $z = 2A\cos\left(\dfrac{kx + ky - 2\omega t}{2}\right)\cos\left[\dfrac{k(x-y)}{2}\right]$

$\therefore$ Amplitude of resultant wave $= 2A\cos\left[\dfrac{k(x-y)}{2}\right]$

For resultant intensity to be zero

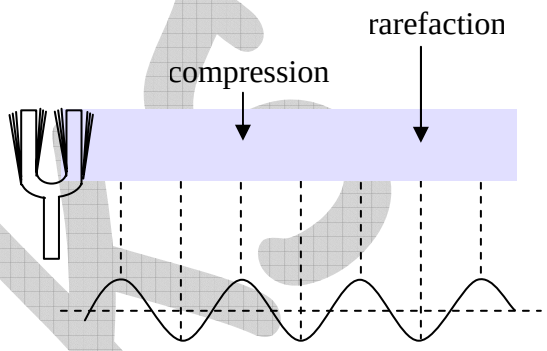
$2A\cos\left[\dfrac{k(x-y)}{2}\right] = 0$ or $(x-y) = \dfrac{(2n+1)\pi}{k}$ where $n = 0, 1, 2, ...$

4.3 Longitudinal Waves

In a longitudinal wave, particles of the medium are displaced in a direction parallel to energy transport.

Longitudinal waves include sound waves, seismic P-waves, ultrasound wave etc.

Sound waves in air (and any fluid medium) are longitudinal waves because particles of the medium through which the sound is transported vibrate parallel to the direction that the sound wave moves. The sound wave traveling through air will cause a sinusoidal pressure variation in the air. The air motion which accompanies the passage of the sound wave will be back and forth in the direction of the propagation of the sound. These back and forth vibrations are imparted to adjacent neighbors by particle-to-particle interaction. Other surrounding particles begin to move rightward and leftward, thus sending a wave to the right. Since air molecules (the particles of the medium) are moving in a direction that is parallel to the direction that the wave moves, the sound wave is referred to as a longitudinal wave. The result of such longitudinal vibrations is the creation of **compressions** and **rarefactions** within the air.



Longitudinal Wave as a Pressure Wave

The displacement equation for plane wave travelling in the $+x$ direction with velocity v is $y = a\sin\frac{2\pi}{\lambda}(vt - x)$

If p is the instantaneous change in pressure at a point x, then $p = -K\frac{\partial y}{\partial x}$

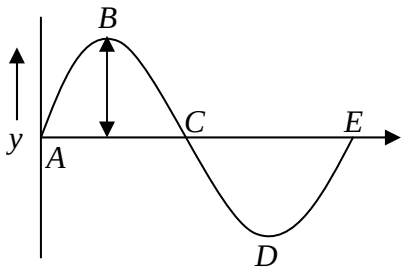
Where, K is the Bulk modulus of elasticity of the medium

$$p = \frac{2\pi Ka}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x)$$

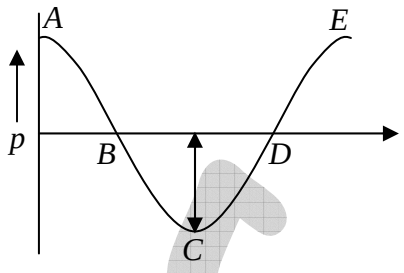
Longitudinal wave velocity in a medium of density ρ is $v = \sqrt{\frac{K}{\rho}}$

Thus $p = \frac{2\pi v^2 \rho a}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) = p_0 \cos \frac{2\pi}{\lambda}(vt - x)$

This is the pressure equation of longitudinal wave.



Displacement curve



Pressure curve

Velocity of the Longitudinal Waves

(i) Velocity of the Longitudinal Waves in solid

When a metallic rod is excited longitudinally by stroking it lengthwise, its cross-sections oscillate to and fro in the direction of the length of the rod and longitudinal waves are set up. The velocity of these waves depends upon the density and elastic modulus of the material. If Y is the young modulus and ρ is the density of the material, than the velocity of longitudinal wave is $v = \sqrt{\frac{Y}{\rho}}$

The velocity of sound in a rod of Nickel $Y = 21.4 \times 10^{10}\ Nm^{-2}$ and $\rho = 8.9 \times 10^{3}\ kgm^{-3}$ is

$$v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{21.4 \times 10^{10}}{8.9 \times 10^{3}}} = 4900\ ms^{-1}$$

Thus the speed of sound in solids is quite high.

(ii) Velocity of the Longitudinal Waves in Gas

When longitudinal wave such as sound wave travel in gas, its particle oscillate in the direction of wave propagation. The result of such longitudinal vibrations is the creation of compressions and rarefactions within the air. Therefore pressure varies from particle to particle in the gas. If K is the Bulk modulus of the gas and ρ is the density, then the velocity of the sound wave in gas is $v = \sqrt{\frac{K}{\rho}}$

The Bulk modulus K is defined as $K = \frac{\text{stress}}{\text{strain}} = -\frac{p}{\frac{dy}{dx}}$

The negative sign indicate that when pressure increases, the volume decreases.

Newton's Correction

Newton assumed that the propagation of sound wave in a gas is an isothermal process. Thus for a given mass of gas at pressure P and volume V, we have from Boyle's Law.

$PV = \text{constant}$ (isothermal change)

On differentiating we get

$$PdV + VdP = 0 \Rightarrow P = -\frac{dP}{\frac{dV}{V}} = \frac{\text{Volume stress}}{\text{Volume strain}} = K\left(\text{Bulk modulus}\right)$$

Thus the velocity of the sound in gas become $v = \sqrt{\frac{P}{\rho}}$

Newton's result did not produce exact result of the sound velocity. As if we substitute the values of standard temperature and pressure (STP), density $\rho = 1.29\ kgm^{-3}$ and Pressure $P = 0.76\ mm$ of $Hg = 0.76 \times \left(13.6 \times 10^3\right) \times 9.8\ Nm^{-2}$ .We get $v = 280\ m/\sec$ while the experimental value is $332\ m/s$. Newton was unable to explain this discrepancy. Thus another correction was made by Laplace in the expression.

Laplace Correction

Laplace suggested that propagation of sound wave in air is an adiabatic process and not an isothermal as pointed by Newton's. According to Laplace, gases are bad conductor of heat. Since the compression and rarefaction occurs very fast, therefore the heat developed at compression can-not go into surrounding and no heat from surrounding comes to rarefaction. Hence, the air in the path of sound wave suffers temperature change, or we can say that the conditions are adiabatic.

Thus the relation between pressure and volume for adiabatic system is $PV^{\gamma} = \text{constant}$, where γ is the adiabatic factor and defined as ratio heat capacity at constant pressure and volume (i.e. $\gamma = \frac{C_p}{C_v}$). Now differentiating the $PV^{\gamma} = \text{constant}$, we get

$$P\gamma V^{\gamma-1}dV + V^{\gamma}dP = 0 \quad \Rightarrow \gamma P = -\frac{dP}{\frac{dV}{V}} = \frac{\text{Volume stress}}{\text{Volume strain}} = K\left(\text{Bulk modulus}\right)$$

Thus the velocity of sound in gas is $v = \sqrt{\frac{\gamma P}{\rho}}$

This is Laplace corrected sound velocity. By taking, $\gamma = 1.41$ for air, velocity of sound at STP is $v = 331\ m/s$, which is close to the actual value.

Effect of Temperature on Sound Velocity in Gases

The density of a gas is $\rho = \frac{M}{V} = \frac{M}{RT/P} \Rightarrow \frac{P}{\rho} = \frac{RT}{M}$

Where, R is the gas constant. The velocity of sound can be written as $v = \sqrt{\frac{\gamma RT}{M}}$

Thus the velocity of sound in a gas

(a) is independent of pressure or density,

(b) is proportional to the square root of the absolute temperature of the gas

(c) is inversely proportional to square root of the molecular weight of the gas

(d) and depends on the value of γ of a gas, i.e. whether gas is mono-atomic, diatomic etc.

Example: The air column in a pipe closed at one end is made to vibrate in its second overtone by a tuning fork of frequency $440\,Hz$. The speed of sound in air is $330\,ms^{-1}$. End corrections may be neglected. Find the length L of the air column.

Solution:

At second overtone $L=\frac{5\lambda}{4}$, $\Rightarrow L=\frac{5}{4}\times\frac{\text{velocity}}{\text{frequency}}$

$$\Rightarrow L=\frac{5}{4}\times\frac{330}{440}=\frac{5}{4}\times\frac{3}{4}\Rightarrow L=\frac{15}{16}m$$

Example: The first overtone of an open organ pipe beats with the first overtone of a closed organ pipe with a beat frequency of $2.2\,Hz$. The fundamental frequency of the closed organ pipe is $110\,Hz$. Find the lengths of the pipes. (speed of sound is $330\,m/s$)

Solution: Frequency of the first overtone for a open pipe is $f_O=3\frac{v}{4l_O}$

Frequency of the first overtone of a closed pipe is $f_C=2\frac{v}{2l_C}$

Fundamental frequency of the closed pipe

$$f=\frac{v}{2l_c}=110\,Hz\Rightarrow l_C=\frac{330}{2\times110}=1.5\,m$$

First case: let $f_C>f_O$; $\therefore f_C-f_O=2.2\ \Rightarrow 220-\frac{3\times330}{4l_O}=2.2\Rightarrow l_O=1.14\ m$

Second: Let $f_0>f_C$; $\therefore f_O-f_C=2.2\ \Rightarrow\frac{3\times330}{4l_O}-220=2.2\ \Rightarrow l_O=1.11\ m$.

4.4 Transverse Wave

Velocity of Transverse Waves in a Stretched String

When a infinitely thin, uniform and perfectly flexible string is displaced at any point and left, transverse waves starts moving from that point along both sides of the string with certain velocity. The whole string vibrates at right angle to its length and the tension in the string provides the restoring force necessary for vibration.

Transverse wave velocity in a stretched string is $v = \sqrt{\frac{T}{\mu}}$, where T is the tension in the string and μ is mass per unit length.

Velocity expression shows that it is independent of wavelength i.e. waves of all wavelength travels with the same velocity in a given stretched string.

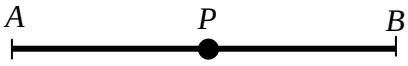
Let us consider a string of mass 1.0 gm^{-1} is stretched with a force of $10N$. The velocity of transverse wave in string is $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{10}{10^{-3}}} = 100\ ms^{-1}$

Normal Modes of Oscillations of a Stretched String

Consider a string of length l, clamped rigidly at both ends A and B. When string is plucked at P, identical wave travel along PA and PB, which are reflected at A and B. The waves travelling in opposite direction will superimpose and produce standing wave.

The equation of these two waves are

$$y_1 = a\sin\frac{2\pi}{\lambda}(vt - x) \text{ and } y_2 = a'\sin\frac{2\pi}{\lambda}(vt + x)$$



The equation of the resultant wave is $y = a\sin\frac{2\pi}{\lambda}(vt - x) + a'\sin\frac{2\pi}{\lambda}(vt + x)$

The two boundary conditions are (i) $y = 0$ at $x = 0$ and (ii) $y = 0$ at $x = l$

The first boundary condition gives $a = -a'$, hence $y = -2a\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$

This represents a stationary wave.

The second boundary condition gives, $-2a\sin\frac{2\pi l}{\lambda}\cos\frac{2\pi vt}{\lambda}=0$

This must be true for all values of t, thus

$\sin\frac{2\pi l}{\lambda}=0\Rightarrow\frac{2\pi l}{\lambda}=n\pi\Rightarrow\lambda=\frac{2l}{n}$ where $n=0,1,2,3....$

The allowed frequencies in such medium is $f=\frac{v}{\lambda}=\frac{nv}{2l}$

Since the velocity of transverse string is $v=\sqrt{\frac{T}{\mu}}$

Thus the allowed frequency become $f=\frac{n}{2l}\sqrt{\frac{T}{\mu}}$

This shows that the string sustain vibration of certain discrete frequencies $(n=1,2,3....)$. These frequencies are 'Normal frequencies' or 'Normal Modes' of the string.

The frequency corresponding to $n=1$ is called Fundamental Frequency of the string, while those corresponding to $n=2,3,4....$ are called first, second, third overtones. The frequencies are in harmonic series $\left(f_1:f_2:f_3:....=1:2:3....\right)$. Thus they are also called Harmonics of the string.

$f_1=\frac{1}{2l}\sqrt{\frac{T}{\mu}}$ (Fundamental Mode or 1st Harmonic)

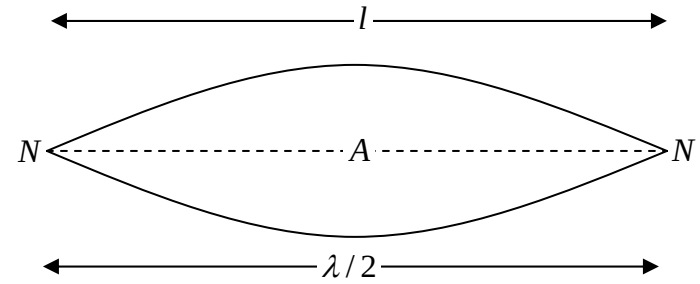
$f_2=\frac{2}{2l}\sqrt{\frac{T}{\mu}}$ (1st overtone or 2nd Harmonic)

$f_3=\frac{3}{2l}\sqrt{\frac{T}{\mu}}$ (2nd overtone or 3rd Harmonic)

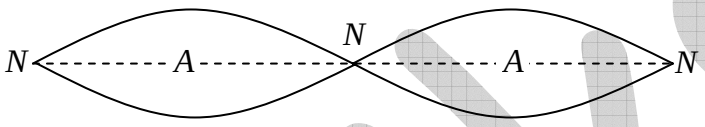
The allowed frequencies define the length of the string as an exact number of half wavelengths.

$$l=n\frac{\lambda}{2}=\frac{\lambda}{2},\frac{2\lambda}{2},\frac{3\lambda}{2},.....$$

These modes are shown in the following figures:



n =1
Fundamental Mode or
1st Harmonic



n =2
1st Overtone or
2nd Harmonic



n =3
2nd Overtone or
3rd Harmonic

fiziks

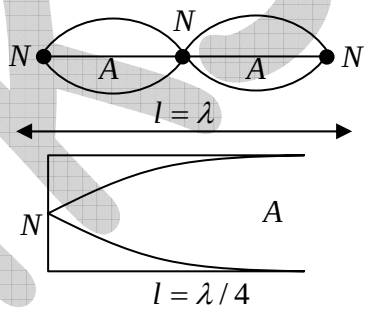
Example: A string $25\,cm$ long and having a mass of $2.5\,g$ is under tension. A pipe closed at one end is $40\,cm$ long. When the string is set vibrating in its first overtone and the air in the pipe in its fundamental frequency 8 beats per second are heard. It is observed that decreasing the tension in the string decreases beat frequency. If the speed of sound in air is $320\,m/s$ find the tension in the string.

Solution: For a vibrating string, velocity of wave $v=\sqrt{\frac{T}{\mu}}$

$$\mu=\frac{\text{Mass}}{\text{length}}=\frac{2.5\times10^{-3}}{2.5\times10^{-2}}=\frac{1}{100}\frac{\text{kg}}{\text{m}}=0.01\frac{\text{kg}}{\text{m}}$$

When the string is set into its first overtone,

$$\lambda=25\,cm=25\times10^{-2}\,m=0.25\,m$$



$\therefore$ Frequency $=\frac{v}{\lambda}$ or $f=\frac{1}{0.25}\sqrt{\frac{T}{0.01}}$ (i)

For pipe closed at one end,

Fundamental frequency$=\frac{\text{velocity}}{\lambda}$

$$f'=\frac{320}{4\times0.4}=200\,Hz \qquad \text{(ii)}$$

Beat frequency $=(f-f')$ or $(f'-f)$

$\therefore f$ depends upon tension T, $\therefore f$ decreases when T, decreases

$\therefore$ Beat frequency (f_b) decreases if $f_b=f-f'$ and (f_b) increases if $f_b=f'-f$.

According to condition given in question, f_b decreases.

$$\therefore f_b=f-f'\Rightarrow 8=\frac{1}{0.25}\sqrt{\frac{T}{0.01}}-200 \text{ or } 208\times0.25=\sqrt{\frac{T}{0.01}} \text{ or } (52)^2\times0.01=T$$

$$\Rightarrow T=27.04\,N.$$

4.5 Phase Velocity

The phase velocity of a wave is the rate at which the phase of the wave propagates in space. This is the velocity at which the phase of any one frequency component of the wave travels. For such a component, any given phase of the wave will appear to travel at the phase velocity.

Consider a monochromatic wave with angular frequency $\omega = 2\pi\nu$ (where ν is the frequency) travelling in $+ve$ x-direction is given by

$$y = a\sin(\omega t - kx)$$

Consider a point on the sine wave at a particular instant of time the wave equation can be written as $y = a\sin\omega t$.

After certain time say t_0 if the point is displaced by a distance x the equation is given by $y = a\sin\left[\omega(t - t_0)\right]$

Velocity of the point $v = \dfrac{x}{t_0}$ or $t_0 = \dfrac{x}{v}$

Therefore $y = a\sin\left[\omega\left(t - \dfrac{x}{v}\right)\right] = a\sin\left[\omega t - \left(\dfrac{\omega}{v}\right)x\right] = a\sin(\omega t - kx)$ where $k = \dfrac{\omega}{v}$

Where k is defined as the propagation constant or the wave number which is also defined as $k = \dfrac{2\pi}{\lambda}$. '$v$' mentioned above as stated earlier is associated with the individual wave and hence it is nothing but phase velocity.

Thus phase velocity $v_p = \dfrac{\omega}{k}$

4.6 Group Velocity

A number of waves of different frequencies, wavelengths and velocities may be superposed to form a group. Waves rarely occur as single monochromatic components; a white light pulse consists of an infinitely fine spectrum of frequencies and the motion of such a pulse would be described by its group velocity. Such a group would, of course, 'disperse' with time because the wave velocity of each component would be different in all media except free space. Only in free space would it remain as white light. Its importance is that it is the velocity with which the energy in the wave group is transmitted. For a monochromatic wave the group velocity and the wave velocity are identical.

We begin by considering a group which consists of two components of equal amplitude a but frequencies ω_1 and ω_2 which differ by a small amount.

Their separate displacements are given by $y_1 = a\cos(\omega_1 t - k_1 x)$ and $y_2 = a\cos(\omega_2 t - k_2 x)$.

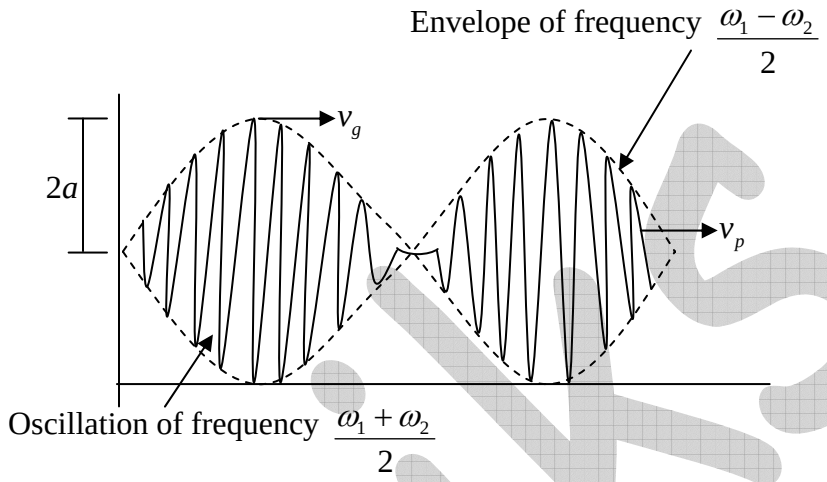
Superposition of amplitude and phase gives

$$y = y_1 + y_2 = 2a\cos\left[\frac{(\omega_1 - \omega_2)}{2}t - \frac{(k_1 - k_2)}{2}x\right]\cos\left[\frac{(\omega_1 + \omega_2)}{2}t - \frac{(k_1 + k_2)}{2}x\right]$$

a wave system with a frequency $\frac{(\omega_1 + \omega_2)}{2}$ which is very close to the frequency of either component but with a maximum amplitude of $2a$, modulated in space and time by a very slowly varying envelope of frequency $\frac{(\omega_1 - \omega_2)}{2}$ and wave number $\frac{(k_1 - k_2)}{2}$.

Then $\frac{(\omega_1 - \omega_2)}{2}t - \frac{(k_1 - k_2)}{2}x = \text{constant} \Rightarrow v_g = \frac{dx}{dt} = \frac{\omega_1 - \omega_2}{k_1 - k_2}$

Figures shows that the faster oscillation occurs at the average frequency of the two components $\frac{(\omega_1+\omega_2)}{2}$ and the slowly varying group envelope has a frequency $\frac{(\omega_1-\omega_2)}{2}$, half the frequency difference between the components



Wave Groups and Group Velocity

Suppose now that the two frequency components of the last section have different phase velocities so that $\frac{\omega_1}{k_1} \neq \frac{\omega_2}{k_2}$. The velocity of the maximum amplitude of the group; that is, the group velocity

$$\frac{(\omega_1-\omega_2)}{(k_1-k_2)} = \frac{\Delta\omega}{\Delta k}$$

is now different from each of these velocities; the superposition of the two waves will no longer remain constant and the group profile will change with time.

A medium in which the phase velocity is frequency dependent ($\frac{\omega}{k}$ not constant) is known as a dispersive medium and a dispersion relation expresses the variation of ω as a function of k. If a group contains a number of components of frequencies which are nearly equal the original expression for the group velocity is written

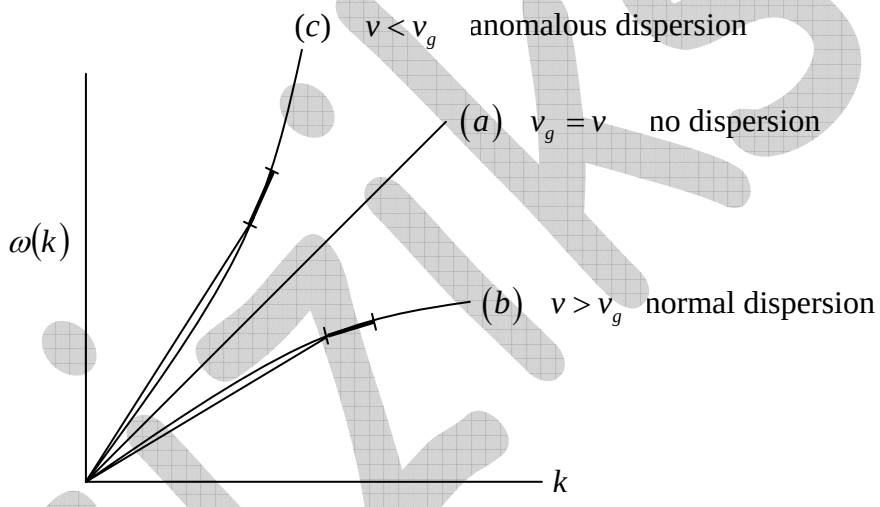
$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

The group velocity is that of the maximum amplitude of the group so that it is the velocity with which the energy in the group is transmitted. Since $\omega = kv$, where v is the phase velocity, the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(kv) = v + k\frac{dv}{dk} = v - \lambda\frac{dv}{d\lambda} \quad \text{where } k = \frac{2\pi}{\lambda}.$$

If $\frac{dv}{d\lambda}$ is positive, then $v > v_g$ (Normal dispersion)

If $\frac{dv}{d\lambda}$ is negative, then $v < v_g$ (Anomalous dispersion).



Above figure gives curves illustrating dispersion relations: (a) a straight line representing a non-dispersive medium, $v = v_g$; (b) a normal dispersion relation where the gradient $v = \frac{\omega}{k} > v_g = \frac{d\omega}{dk}$; (c) an anomalous dispersion relation where $v < v_g$.

The three curves of above Figure represent

- A non-dispersive medium where $\frac{\omega}{k}$ is constant, so that $v = v_g$, for instance free space behaviour towards light waves.
- A normal dispersion relation $v > v_g$.
- An anomalous dispersion relation $v < v_g$.

Example: What are the values of phase velocity and group velocity respectively of the de-Broglie wave describing a free electron with velocity v?

Solution: According to de-Broglie, every moving object is associated with a wave, known as de-Broglie wave. If m is mass of the object moving with velocity v then associated wavelength λ is given as $\lambda = \frac{h}{mv}$ where h is Planck's constant

The velocity of wavefront of constant phase is called phase velocity and is equal to the velocity of the wave

Hence, phase velocity $= \lambda\nu$, by energy equation $mc^2 = h\nu \Rightarrow \nu = \frac{mc^2}{h}$

Thus Phase velocity $= \lambda \times \nu = \frac{h}{mv} \times \frac{mc^2}{h} = \frac{c^2}{v}$

Now, the group velocity is given as $v_g = \frac{d\omega}{dk}$

$\because \omega = 2\pi\nu = \frac{2\pi mc^2}{h}$ and $k = \frac{2\pi}{\lambda} = \frac{2\pi mv}{h}$ where $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$

Thus, $\omega = \frac{2\pi m_0 c^2}{h\sqrt{1 - v^2/c^2}}$ and $k = \frac{2\pi m_0 v}{h\sqrt{1 - v^2/c^2}}$

By differentiation we get,

$\frac{d\omega}{dv} = \frac{2\pi m_0 v}{h\left(1 - v^2/c^2\right)^{3/2}}$ and $\frac{dk}{dv} = \frac{2\pi m_0}{h\left(1 - v^2/c^2\right)^{3/2}}$

$\Rightarrow v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv} = \frac{2\pi m_0 v}{h\left(1 - v^2/c^2\right)^{3/2}} \Big/ \frac{2\pi m_0}{h\left(1 - v^2/c^2\right)^{3/2}} = v$

MCQ (Multiple Choice Questions)

Q1. A transverse wave is described by the equation $y = y_0 \sin 2\pi\left(ft - \frac{x}{\lambda} \right)$. The maximum particle velocity is equal to four times the wave velocity if

(a) $\lambda = \pi \frac{y_0}{4}$ (b) $\lambda = \pi \frac{y_0}{2}$

(c) $\lambda = \pi y_0$ (d) $\lambda = 2\pi y_0$.

Q2. The vibration of a string can be described by the equation

$$y = (0.75\ cm)\left\{\cos\left(\frac{2\pi}{5cm^{-1}}\right)x\right\}\left\{\sin\left((60\pi t\ s^{-1})\right)\right\}$$

The speed of the waves traveling in the string is:

(a) 24 cm/s (b) 0.3 cm/s (c) 450 cm/s (d) 150 cm/s

Q3. A plane progressive wave is traveling with wave vector $\vec{k}$; $\hat{n}$ is the unit vector in the direction of amplitude. If the wave is longitudinal, then which one of the following is correct?

(a) $\hat{n}\times\vec{k} \neq \vec{0}, \hat{n}\cdot\vec{k} = 0$ (b) $\hat{n}\times\vec{k} = \vec{0}, \hat{n}\cdot\vec{k} = 0$

(c) $\hat{n}\times\vec{k} = \vec{0}, \hat{n}\cdot\vec{k} \neq 0$ (d) $\hat{n}\times\vec{k} \neq \vec{0}, \hat{n}\cdot\vec{k} \neq 0$

Q4. An open pipe of length 33 cm resonates to a frequency of 1000 Hz. The mode of vibration is:

(a) fundamental (b) the first harmonic

(c) the second harmonic (d) the fourth harmonic

Q5. An open pipe is suddenly closed at one end with the result that the frequency of second harmonic of the closed pipe is found to be higher by 100 Hz than the fundamental frequency of the open pipe. The fundamental frequency of the open pipe is

(a) 200 Hz (b) 300 Hz (c) 240 Hz (d) 480 Hz

Q6. A pipe of length l_1 closed at one end is kept in a chamber of gas of density ρ_1. A second pipe open at both ends is placed in a second chamber of gas of density ρ_2. The compressibility of both the gases is equal. Calculate the length of the second pipe if frequency of first overtone in both the cases is equal

(a) $\frac{4}{3}l_1\sqrt{\frac{\rho_2}{\rho_1}}$ (b) $\frac{4}{3}l_1\sqrt{\frac{\rho_1}{\rho_2}}$

(c) $l_1\sqrt{\frac{\rho_2}{\rho_1}}$ (d) $l_1\sqrt{\frac{\rho_1}{\rho_2}}$

Q7. An open pipe is in resonance in 2nd harmonic with frequency f_1. Now one end of the tube is closed and frequency is increased to f_2 such that the resonance again occurs in n^{th} harmonic. Choose the correct option

(a) $n=3,\ f_2=\frac{3}{4}f_1$ (b) $n=3,\ f_2=\frac{5}{4}f_1$

(c) $n=5,\ f_2=\frac{3}{4}f_1$ (d) $n=5,\ f_2=\frac{5}{4}f_1$.

Q8. A tube closed at one end and containing air when excited, produces the fundamental note of frequency $512\,Hz$. If the tube is open at both ends the fundamental frequency that can be excited is (in Hz)

(a) 1024 (b) 512 (c) 256 (d) 128.

Q9. An organ pipe P_1 closed at one end vibrating in its first harmonic and another pipe P_2 open at both ends vibrating in its third harmonic are in resonance with given tuning fork. The ratio of the length of P_1 to that of P_2 is

(a) $\frac{8}{3}$ (b) $\frac{3}{8}$ (c) $\frac{1}{6}$ (d) $\frac{1}{3}$.

Q10. The extension in a string, obeying Hooke's law, is x. The speed of sound in the stretched string is v. If the extension in the string is increased to $1.5x$, the speed of sound will be

(a) $1.22v$ (b) $0.61v$ (c) $1.50v$ (d) $0.75v$.

Q11. Two vibrating strings of the same material but lengths L and $2L$ have radii $2r$ and r respectively. They are stretched under the same tension. Both the strings vibrate in their fundamental nodes, one of length L with frequency f_1 and the other with frequency f_2. The ratio $\frac{f_1}{f_2}$ is given by

(a) 2 (b) 4 (c) 8 (d) 1.

Q12. The ends of a stretched wire of length L are fixed at $x = 0$ and $x = L$. In one experiment, the displacement of the wire is $y_1 = A\sin\left(\frac{\pi x}{L}\right)\sin\omega t$ and energy is E_1 and in another experiment its displacement is $y_2 = A\sin\left(\frac{2\pi x}{L}\right)\sin 2\omega t$ and energy is E_2. Then

(a) $E_2 = E_1$ (b) $E_2 = 2E_1$

(c) $E_2 = 4E_1$ (d) $E_2 = 16E_1$

Q13. If three segments of a wire having fundamental frequency of vibrations n_1, n_2 and n_3 respectively are joined together in series and allowed to vibrate under the same tension, the fundamental frequency of vibration of the whole wire would be n. What is the relationship among these frequencies?

(a) $n = n_1 + n_2 + n_3$ (b) $\frac{1}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$

(c) $n = \frac{(n_1 + n_2 + n_3)}{3}$ (d) $n^{1/2} = n_1^{1/2} + n_2^{1/2} + n_3^{1/2}$

Q14. A long horizontal string of linear density $0.24\ kg/m$ has a tension of $96\ N$. If it is attached to a bar which oscillates in the vertical direction with amplitude of $\pm\ 0.65\ cm$ at a frequency of $100\ Hz$, then the wave on the string will be:

(a) $0.013\sin\left(\frac{\pi}{0.2}x - 200\pi t\right)$ (b) $0.065\sin\left(\frac{\pi}{0.1}x - 200\pi\ \ t\right)$

(c) $0.013\sin\left(\frac{\pi}{0.2}x + 200\pi t\right)$ (d) $0.0065\sin\left(\frac{\pi}{0.1}x - 200\pi t\right)$

Q15. One end of a string is connected to an electrically maintained vibrating bar while the other end is made to pass over a frictionless pulley. The free end is then loaded so that the string is under tension T_1 . The string vibrates in x_1 segments as the bar vibrates in transverse mode. If the tension is changed to T_2 the string vibrates in x_2 segments. Then:

(a) $x_1\sqrt{T_1} = x_2\sqrt{T_2}$ (b) $x_2\sqrt{T_1} = x_1\sqrt{T_2}$

(c) $T_1\sqrt{x_2} = T_2\sqrt{x_1}$ (d) $T_1\sqrt{x_1} = T_2\sqrt{x_2}$

Q16. If waves in an ocean travel with a phase velocity $v_p = A\lambda^{-1/2}$, then the group velocity v_g of the wave packet' of these waves is

(a) $v_g = (3/2)v_p$ (b) $v_g = (5/2)v_p$

(c) $v_g = (2/3)v_p$ (d) $v_g = (2/5)v_p$

Q17. If the dispersion relation for electromagnetic waves of wave vector k in the ionosphere is

$$\omega^2 = \omega_0^2 + c^2k^2$$

then the group velocity v_g and phase velocity v_p are related as:

(a) $v_g = c^2v_p$ (b) $v_p = c^2v_g$

(c) $v_g v_p = c^2$ (d) $v_p - v_g = c^2$

MSQ (Multiple Select Questions)

Q18. A transverse sinusoidal wave of amplitude a, wavelength λ and frequency f is travelling on a stretched string. The maximum speed of any point on the string is $\frac{v}{10}$, where v is the speed of propagation of the wave. If $a = 10^{-3} m$ and $v = 10 ms^{-1}$, then λ and f are given by

(a) $\lambda = 2\pi \times 10^{-2} m$ (b) $\lambda = 10^{-3} m$

(c) $f = \frac{10^3}{2\pi} Hz$ (d) $f = 10^4 Hz$.

Q19. A wave equation which gives the displacement along the y-direction is given by $y = 10^{-4} \sin(60t + 2x)$ where x and y are in meter and t time in second. This represents a wave

(a) Of wavelength π m

(b) Of frequency $\left(\frac{30}{\pi}\right)$ hertz

(c) Travelling with a velocity of $30 m/s$

(d) Of amplitude $10^{-4} m$ in travelling along the negative x- direction.

Q20. A wave is represented by the equation $y = A\sin\left(10\pi x + 15\pi t + \frac{\pi}{3}\right)$ where x is in meter and t is in second. The expression represents

(a) a wave travelling in the positive x- direction with a velocity $1.5 m/s$.

(b) a wave travelling in the negative x -direction with velocity $1.5 m/s$.

(c) a wave travelling in the negative x -direction having a wavelength $0.2 m$

(d) a wave travelling in the positive x-direction having a wavelength $0.2 m$.

Q21. A wave disturbance in a medium is described by

$y(x,t) = 0.02\cos\left(50\pi t + \frac{\pi}{2}\right)\cos(10\pi x)$, where x and y are in metre and t is in second.

(a) A node occurs at $x = 0.15\,m$ (b) An antinode occurs at $x = 0.3\,m$

(c) The speed of wave is $5\,ms^{-1}$ (d) The wavelength is $0.2\,m$.

Q22. A travelling wave has the frequency ν and the amplitude A. Then which of the following statements are true

(a) Amplitude of particle velocity is $2\pi\nu A$

(b) Amplitude of particle velocity is νA

(c) Amplitude of particle acceleration is $4\pi^2\nu^2 A$

(d) Amplitude of particle acceleration is $4\pi^2\nu^2 A^2$

Q23. An air column in a pipe, which is closed at one end, will be in resonance with a vibrating tuning fork of frequency $264\,Hz$ if the length of the column in cm is (speed of sound is $330\ m/s$)

(a) 31.25 (b) 62.50 (c) 93.75 (d) 125.

Q24. Velocity of Sound in air is $320\,m/s$. A pipe closed at one end has a length of $1m$. Neglecting end corrections, the air column in the pipe can resonate for sound of frequency

(a) $80\,Hz$ (b) $240\,Hz$ (c) $320\,Hz$ (d) $400\,Hz$.

NAT (Numerical Answer Type)

Q25. A plane progressive wave of frequency $2.5\,Hz$, amplitude 2.5×10^{-5} and initial phase zero propagates along the negative x - direction with a velocity of $300\,m/s$. At any instant the phase difference between the oscillations at two points $6\,m$ apart along the line of propagation is …….(in radians)

Q26. The displacement y of a particle executing periodic motion is given by

$$y = 4\cos^2\left(\frac{1}{2}t\right)\sin\left(1000t\right)$$

This expression may be considered to be a result of the superposition of .....independent harmonic motions.

Q27. Sound waves of frequency $660\,Hz$ fall normally on a perfectly reflecting wall. The shortest distance from the wall at which the air particles have maximum amplitude of vibration is _____metre. (Speed of sound is $330\ m/s$)

Q28. An open organ pipe has a fundamental frequency of $300\,Hz$. The first overtone of this organ pipe is same as the first overtone of a closed organ pipe. The length of the closed organ pipe is (assume velocity of sound in air as $330\ m/s$) .....cm

Q29. Two organ pipes closed at one end when blown simultaneously produce $4\,beat/s$. If the length of the shorter pipe is $0.75\ m$, the length of the other will be ..... cm
(Velocity of sound = 330 m/s)

Q30. A string $1\ m$ long is drawn by a $300\ Hz$ vibrator attached to its end. The string vibrates in three segments. The speed of transverse waves in the string is equal to_____ m/s

Q31. A $1m$ long string of mass $4.9\times10^{-4}\ kg$ is held under a tension of $19.6\ N$. If the string vibrates in one segment, then the frequency of vibration will be .....Hz

Q32. A wire under tension vibrates with a fundamental frequency of $460\ Hz$. If the length of the wire is halved, radius made twice and is made to vibrate under one-fourth the tension, then the fundamental frequency will now be equal to..... Hz

Solution-MCQ (Multiple Choice Questions)

Ans. 1: (b)

Solution: (b): $y = y_0 \sin 2x\left(ft - \frac{x}{\lambda} \right) \Rightarrow \frac{dy}{dt} = 2\pi f y_0 \cos 2\pi \left(ft - \frac{x}{\lambda} \right) \Rightarrow \left(\frac{dy}{dt} \right)_{\max} = 2\pi f y_0$

$\therefore$ Maximum particle velocity $= 2\pi f y_0$ and wave velocity $v = f\lambda$

$$\because 2\pi f y_0 = 4 \times (f\lambda) \Rightarrow \lambda = \frac{\pi y_0}{2}$$

Ans. 2: (d) Solution: $y = (0.75cm)\cos\frac{2\pi}{5} x \sin 60\pi t$

Comparing this with the standard equation of standing wave

$y = 2a\cos\frac{2\pi x}{\lambda}\sin\omega t$ or $y = 2a\cos kx \sin\omega t$

we have, $k = \frac{2\pi}{5}, \omega = 60\pi$

Velocity of wave $v = \frac{\omega}{k} \Rightarrow v = \left(\frac{60\pi}{2\pi/5} \right) cm/s = 150\ cm/s$

Ans. 3: (c)

Ans. 4: (c)

Solution: The frequency in open pipe is given as $\nu = \frac{v}{2L} \Rightarrow \nu = \frac{330}{33\times10^{-2}\times 2} = 500\ Hz$

Since, frequency (1000) is double of fundamental frequency (500). Hence, second harmonic take place.

Ans. 5: (a)

Solution: (a): For open pipe, in fundamental mode, $l = \frac{\lambda_1}{2}$

$\therefore$ Frequency $= \frac{v}{\lambda_1} = \frac{v}{2l} \Rightarrow f_1 = \frac{v}{2l}$

For closed pipe, vibrating in third harmonic, $l = \frac{3\lambda_2}{4}$

$\therefore$ Frequency $= \frac{v}{\lambda_2} = \frac{3v}{4l}$ or $f_2 = \frac{3v}{4l}$

$\because f_2 - f_1 = 100 \Rightarrow \frac{3v}{4l} - \frac{v}{2l} = 100 \Rightarrow v = 400l \therefore f_1 = \frac{v}{2l} = \frac{400l}{2l} = 200\,Hz$

$\therefore$ Fundamental frequency of open pipe $= 200\,Hz$

Ans. 6: (b)

Solution: (b): The first pipe is closed at one end let its length be l_1.

$\therefore$ For first overtone, $l_1 = \frac{3\lambda_1}{4} \therefore \lambda_1 = \frac{4l_1}{3}$

$\therefore$ Frequency $= \frac{\text{Velocity in gas}}{\lambda_1} = \frac{1}{\lambda_1}\sqrt{\frac{E}{\rho_1}} \Rightarrow f_1 = \frac{3}{4l_1}\sqrt{\frac{E}{\rho_1}}$

The second pipe is open at both end let its length be l_2.

$\therefore$ For first overtone, $l_2 = \lambda_2$

$\therefore$ Frequency $= \frac{v_2}{\lambda_1} = \frac{1}{\lambda_2}\sqrt{\frac{E}{\rho_2}} \Rightarrow f_2 = \frac{1}{l_2}\sqrt{\frac{E}{\rho_2}}$

Equate the two frequencies, $\frac{1}{l_2}\sqrt{\frac{E}{\rho_2}} = \frac{3}{4l_1}\sqrt{\frac{E}{\rho_1}} \Rightarrow l_2 = \frac{4l_1}{3}\sqrt{\frac{\rho_1}{\rho_2}}$

Ans. 7: (d) Solution: (d): Initially, the pipe is open at both ends.

For second harmonic, $\lambda = l$ $\therefore$ Frequency $f_1 = \frac{v}{\lambda} = \frac{v}{l}$

Finally, the pipe is closed at one end.

For n^{th} harmonic, $\lambda = \frac{4l}{n}$ where n is an odd number.

$\therefore$ Frequency $f_2 = \frac{v}{\lambda} \Rightarrow f_2 = \frac{nv}{4l}$

$f_2 = \frac{n}{4} f_1$. Here n is an odd number.

If $n = 3$, $f_2 < f_1$. This is not acceptable. If $n = 5$, $f_2 = \frac{5}{4} f_1$. This is acceptable.

Ans. 8: (a) Solution: (a): For tube closed at one end, Fundamental frequency $f = \frac{v}{\lambda} = \frac{v}{4l}$

For tube open at both ends, Fundamental frequency $f' = \frac{v}{\lambda'} = \frac{v}{2l}$

$\therefore \frac{f'}{f} = \frac{v}{2l} \times \frac{4l}{v} = 2 \Rightarrow f' = 2f \Rightarrow f' = 1024\,Hz$.

Ans. 9: (c) Solution: for pipe closed at one end, Frequency of first harmonic, $f_1 = \frac{v}{4l_1}$

For pipe open at both ends, frequency of third harmonic $f_2 = \frac{3v}{2l_2}$

$\because f_1 = f_2 \Rightarrow \frac{v}{4l_1} = \frac{3v}{2l_2} \Rightarrow \frac{l_1}{l_2} = \frac{1}{6}$

Ans. 10: (a)

Solution: $\therefore v = \sqrt{\frac{T}{\mu}}$, where $T =$ tension in string, μ is the linear mass density

The tension in the string is due to the load or force applied to it. The load causes extension of the string. According to Hooke's law, T is proportional to x.

$\therefore \frac{v}{v'} = \sqrt{\frac{T}{T'}} \Rightarrow \frac{v}{v'} = \sqrt{\frac{x}{1.5x}} \Rightarrow v' = v \times \sqrt{1.5} \Rightarrow v' = 1.22v$.

Ans. 11: (d)

Solution: For a vibrating string, in fundamental mode, $f = \frac{1}{2l}\sqrt{\frac{T}{\mu}} \quad \because \mu = \frac{\text{Mass}}{\text{length}}$

$\Rightarrow f = \frac{1}{2l}\sqrt{\frac{T}{\pi r^2 \rho}} = \frac{1}{2lr}\sqrt{\frac{T}{\pi\rho}} \qquad \because \mu = \frac{\text{volume} \times \text{density}}{\text{length}} = \frac{\pi r^2 l \rho}{l} = \pi r^2 \rho$

$\therefore \frac{f_1}{f_2} = \left(\frac{l_2}{l_1}\right)\left(\frac{r_2}{r_1}\right)$ as T, ρ are same.For f_1, quantities are L and $2r$, For f_2, quantities are $2L$ and r. $\therefore \frac{f_1}{f_2} = \left(\frac{2L}{L}\right)\left(\frac{r}{2r}\right) \Rightarrow \frac{f_1}{f_2} = 1$.

Ans. 12: (c)

Solution: Energy is proportional to $A^2 f^2$ where A denotes amplitude and f denotes frequency.

$$\therefore \frac{E_2}{E_1} = \left(\frac{A_2}{A_1}\right)^2 \left(\frac{f_2^2}{f_1^2}\right) \quad \text{where } f_2 = \frac{2\omega}{2\pi} \text{ and } f_1 = \frac{\omega}{2\pi}.$$

$$\text{or } \frac{E_2}{E_1} = \left(\frac{A}{A}\right)^2 \left(\frac{2\omega}{2\pi} \times \frac{2\pi}{\omega}\right)^2 \Rightarrow \frac{E_2}{E_1} = 4 \Rightarrow E_2 = 4E_1.$$

Ans. 13: (b)

Solution: If L is length of a wire then fundamental wavelength is given as

$$\frac{\lambda}{2} = L \text{ i.e. } \lambda = 2L \Rightarrow f = \frac{v}{2L}$$

Hence,
$$\left.\begin{aligned} n_1 &= \frac{v}{2L_1} \Rightarrow L_1 = \frac{v}{2n_1}, \\ n_2 &= \frac{v}{2L_2} \Rightarrow L_2 = \frac{v}{2n_2}, \\ n_3 &= \frac{v}{2L_3} \Rightarrow L_3 = \frac{v}{2n_3} \end{aligned}\right\}$$

Now, if n is fundamental frequency of whole wire, then

$$n = \frac{v}{2(L_1 + L_2 + L_3)} \Rightarrow L_1 + L_2 + L_3 = \frac{v}{2n}$$

$$\Rightarrow \frac{v}{2n_1} + \frac{v}{2n_2} + \frac{v}{2n_3} = \frac{v}{2n} \Rightarrow \frac{1}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$$

Ans. 14: (d)

Solution: $m = 0.24\ kg/m$, $T = 96N$, Amplitude, $a = \pm 0.65\ cm$, Frequency $\nu = 100\ Hz$

The velocity of wave $v = \sqrt{\frac{T}{m}} \Rightarrow v = \sqrt{\frac{96}{0.24}} = \sqrt{400} = 20\ m/s$

The equation of wave $y = a\sin 2\pi\left(\nu t - \frac{x}{\lambda}\right)$

$$y = 0.0065\sin(200\pi t - 10\pi x) = 0.0065\sin\left(\frac{\pi}{0.1}x - 200\pi t\right)$$

Ans. 15: (a)

Solution: The formula for frequency $\nu = \frac{p}{2L}\sqrt{\frac{T}{m}}$

$L \rightarrow$ Length, $p \rightarrow$ number of segments, $T \rightarrow$ tension

For a string with constant frequency; $p\sqrt{T} =$ constant

$\Rightarrow \qquad x_1\sqrt{T_1} = x_2\sqrt{T_2}$

Ans. 16: (a)

Solution: The phase velocity $v_p = A\lambda^{-\frac{1}{2}}$ where $\lambda =$ wavelength

Phase velocity $v_p = \frac{\omega}{k} \Rightarrow \frac{\omega}{k} = A\lambda^{-\frac{1}{2}} \Rightarrow \omega = Ak\lambda^{-\frac{1}{2}} = \frac{Ak}{\sqrt{\lambda}}$

$$\Rightarrow \omega = \frac{Ak}{\sqrt{\frac{2\pi}{k}}} \Rightarrow \omega = \frac{Ak^{\frac{3}{2}}}{\sqrt{2\pi}}$$

Group velocity $v_g = \frac{d\omega}{dk} \Rightarrow v_g = \frac{d}{dk}\left(\frac{Ak^{\frac{3}{2}}}{\sqrt{2\pi}}\right) \Rightarrow v_g = \frac{A}{\sqrt{2\pi}}\frac{3}{2}k^{1/2}$

$$\Rightarrow v_g = \frac{A}{\sqrt{2\pi}}\frac{3}{2}\left(\frac{2\pi}{\lambda}\right)^{1/2} \Rightarrow v_g = \frac{3}{2}A\lambda^{-1/2} = \frac{3}{2}v_p$$

Ans. 17: (c)

Solution: We have $\omega^2 = \omega_0^2 + c^2k^2$, differentiate equation, we get $2\omega\, d\omega = 2kc^2 dk$

$$\Rightarrow v_g = \frac{d\omega}{dk} \Rightarrow v_g = \frac{kc^2}{\omega} \text{ and } \omega = kv_p \Rightarrow v_g = \frac{kc^2}{kv_p} \Rightarrow v_g = \frac{c^2}{v_p} \Rightarrow v_g \cdot v_p = c^2$$

MSQ (Multiple Select Questions)

Ans. 18: (a) and (c)

Solution: Maximum speed of particle $= a\omega$ $\because \frac{v}{10} = a\omega \Rightarrow \frac{10}{10} = a \times 2\pi f$

$$f = \frac{1}{2\pi a} = \frac{1}{2\pi \times 10^{-3}} = \frac{1000}{2\pi} = \frac{1000}{2\pi} Hz \Rightarrow \lambda = \frac{v}{f} = \frac{10 \times \pi}{500} = 2\pi \times 10^{-2} m$$

Options (a) and (c) represent correct answers.

Ans. 19: (a), (b), (c) and (d)

Solution: Given: $y = 10^{-4} \sin(60t + 2x) \Rightarrow k = 2,\ a = 10^{-4} m,\ \omega = 60$

$\therefore$ Wavelength $\lambda = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi$ meter, $\therefore$ Frequency $f = \frac{\omega}{2\pi} = \frac{60}{2\pi} = \frac{30}{\pi} Hz$

Velocity $v = f\lambda$ or $v = \frac{30}{\pi} \times \pi$ or $v = 30\, m/s$

Ans. 20: (b) and (c)

Solution: Given equation is $y = A\sin\left(10\pi x + 15\pi t + \frac{\pi}{3}\right)$. Wave travels in negative x-direction.

$$\frac{2\pi}{\lambda} = 10\pi \Rightarrow \lambda = 0.2\, m \text{ and } \frac{2\pi v}{\lambda} = 15\pi \Rightarrow v = \frac{15}{2}\lambda = \frac{15}{2} \times 0.2 = 1.5\ m/s$$

Ans. 21: (a), (b), (c) and (d)

Solution: $y = 0.02\cos\left(50\pi t + \frac{\pi}{2}\right)\cos(10\pi x)$

(a) At node, $y = 0$; Node occurs when $10\pi x = \frac{\pi}{2}, \frac{3\pi}{2} \ldots \Rightarrow\ x = 0.05, 0.15, \ldots..$

(b) At antinode, y is maximum; occurs when $10\pi x = 0, \pi, 2\pi, 3\pi$ or $x = 0, 0.1, 0.2, 0.3, \ldots$

(c) speed of wave is $v = \frac{\omega}{k} = \frac{50\pi}{10\pi} = 5\, ms^{-1}$

(d) Wavelength of the wave $\lambda = \frac{2\pi}{k} = \frac{2\pi}{10\pi} = 0.2\, m$.

Ans. 22: (a), (c)

Solution: Particle velocity amplitude $= A\omega$, Particle acceleration amplitude $= A\omega^2$

where $\omega = 2\pi\nu$

Ans. 23: (a) and (c)

Solution: When a pipe closed at one end vibrates,

$$\therefore \lambda = \frac{4l}{(2n+1)}, \text{ where } (2n+1) \text{ is an odd number .}$$

$$\therefore \text{velocity} = f\lambda \Rightarrow 330 = 264 \times \frac{4l}{(2n+1)}$$

For $n = 0 \Rightarrow l = \frac{330}{264 \times 4} = \frac{15}{48} m = 31.25 cm$, For $n = 1 \Rightarrow l = 3 \times 31.25 = 93.75 cm$

For $n = 2 \Rightarrow l = 5 \times 31.25 = 156.25 cm$

Ans. 24: (a), (b) and (d)

Solution: When a pipe closed at one end vibrates, only odd multiples of fundamental frequencies are present.

$\therefore$ Frequency $f = \frac{(2n+1)v}{4l}$ when $n = 0, 1, 2, ...$

When $n = 0$, fundamental frequency $f_0 = \frac{v}{4l} \Rightarrow f_0 = \frac{320}{4 \times 1} = 80\, Hz$

Other frequencies are $240, 400, 560, Hz$ etc.

NAT (Numerical Answer Type)

Ans. 25: 3.14

Solution: wavelength $\lambda = \dfrac{\text{velocity}(v)}{\text{frequency}(f)} = \dfrac{300}{25} = 12\,m$

Phase difference $\Delta\varphi = \dfrac{2\pi}{\lambda}(\Delta x) \therefore \quad \Delta\phi = \dfrac{2\pi}{12} \times 6 = \pi$

Ans. 26: 3

Solution: $y = 4\cos^2\left(\dfrac{t}{2}\right)\sin(1000t)$

$\therefore y = 4\left(\dfrac{1+\cos t}{2}\right)\sin(1000t) \quad \because \cos t = 2\cos^2\left(\dfrac{t}{2}\right) - 1 \Rightarrow \cos^2\left(\dfrac{t}{2}\right) = \dfrac{1+\cos t}{2}$

$\Rightarrow y = 2(1+\cos t)\sin(1000t) \Rightarrow y = 2\sin(1000t) + 2\cos t \sin(1000t)$

$\Rightarrow y = 2\sin(1000t) + \sin(1001t) + \sin(999t)$

Three independent harmonic motions are superposed for given periodic motion.

Ans. 27: 0.125

Solution: Wall denotes the position of formation of node. Displacement at node is zero. The particles will have maximum amplitude of vibration at neighbouring antinode. Antinode is separated from node by a distance of $\dfrac{\lambda}{4}$.

$\therefore \lambda = \dfrac{v}{f} = \dfrac{330}{660} = 0.5\,m$. $\therefore$ Desired distance $= \dfrac{\lambda}{4} = \dfrac{0.5}{4} = 0.125\,m$.

Ans. 28: 41.3

Solution: If l_1 and l_2 are lengths of open and closed organ pipes then

(i) first overtone of organ pipe $\dfrac{2v}{2l_1}$, (ii) first overtone of closed pipe $\dfrac{3v}{4l_2}$

(iii) Fundamental frequency of open organ pipe $= \dfrac{v}{2l_1} = 300$

$$\Rightarrow \frac{330}{2l_1} = 300 \Rightarrow l_1 = \left(\frac{330}{300\times 2}\right) m = 55cm$$

By second condition $\frac{2v}{2l_1} = \frac{3v}{4l_2} \quad \Rightarrow \frac{1}{l_1} = \frac{3}{4l_2} \Rightarrow l_2 = \frac{3}{4}\times l_1 = \frac{3}{4}\times 55\ cm = 41.3\ cm$

Ans. 29: 78

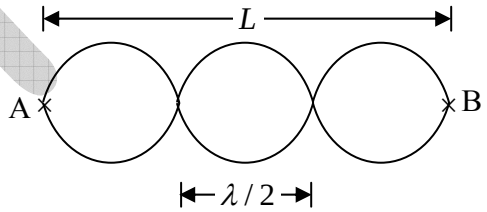
Solution: In closed organ pipe the fundamental frequency is $f = \frac{v}{4L} \Rightarrow f_1 = \frac{v}{4L_1},\ f_2 = \frac{v}{4L_2}$

$$\Rightarrow f_1 - f_2 = \frac{v}{4}\left(\frac{1}{L_1} - \frac{1}{L_2}\right) \Rightarrow 4 = \frac{v}{4}\left[\frac{1}{0.75} - \frac{1}{L_2}\right] \Rightarrow 4 = \frac{330}{4}\left[\frac{4}{3} - \frac{1}{L_2}\right]$$

$$\frac{16}{330} = \frac{4}{3} - \frac{1}{L_2} \Rightarrow \frac{1}{L_2} = \frac{4}{3} - \frac{16}{330} = \frac{440-16}{330} = \frac{424}{330} \Rightarrow L_2 = \frac{330}{424}\ m = 0.778\,m = 78\,cm$$

Ans. 30: 200

Solution: If L is length of the string, then $3\left(\frac{\lambda}{2}\right) = L$

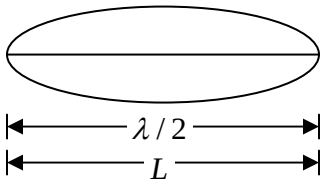


$$\Rightarrow \lambda = \frac{2L}{3} \Rightarrow v = \nu\lambda = \nu\frac{2L}{3} \Rightarrow v = 300\frac{2}{3}\times\times 1 = 200\ m/s$$

Ans. 31: 100

Solution: The velocity of vibration in a wire of mass per unit length m stretched by tension T is given as

$$v = \sqrt{\frac{T}{m}} = \frac{19.6}{4.9\times 10^{-4}} = 2\times 10^2\, m/s$$



Now frequency

$$\nu = \frac{v}{\lambda} = \frac{v}{2L} = \frac{2\times 10^2}{2\times 1} = 100Hz$$

Ans. 32: 230

Solution: Frequency $\nu = \frac{1}{2L}\sqrt{\frac{T}{m}}$ where m is mass per unit length of string of length L under tension T.

$$\nu = \frac{1}{2L}\sqrt{\frac{T}{\pi r^2 d}} \qquad \left(\because m = d\pi r^2\right), \quad d \text{ is linear mass density.}$$

By $\nu^2 \propto \frac{T}{r^2 L^2}$

$$\Rightarrow \qquad \frac{\nu_2^2}{\nu_1^2} = \frac{T_2}{T_1}\left(\frac{r_1}{r_2}\right)^2\left(\frac{L_1}{L_2}\right)^2 = \left(\frac{1}{4}\right)(2)^2\left(\frac{1}{2}\right)^2$$

$$\Rightarrow \qquad \nu_2 = \frac{\nu_1}{2} = 460 \times \frac{1}{2} = 230 \text{ Hz}$$

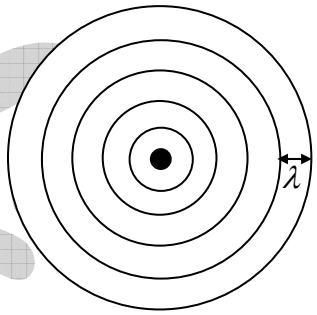
5. The Doppler Effect

Whenever there is a relative motion between an observer and a distant sounding source, the pitch (frequency) of the sound wave as heard by the observer is different from its true pitch. The pitch heard is appeared to be higher than the true pitch if the distance between the source and the observer is decreasing, and vice versa.

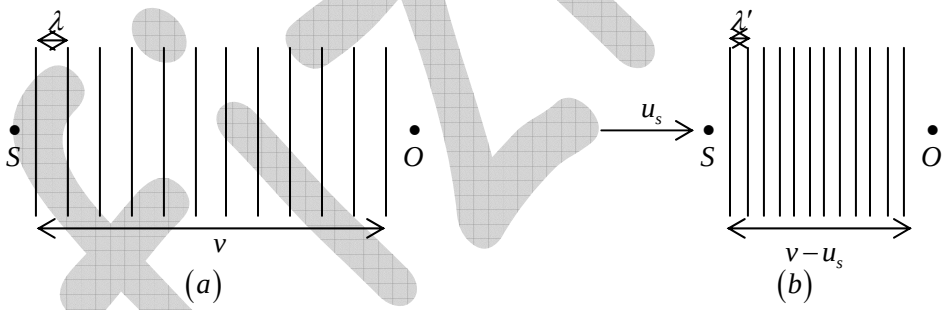
5.1 Doppler Effect in Sound Waves

Sound Waves from a Source at Rest

A source at rest emitting a steady note generates circular wave crests. The concentric circles represent wave crests generated by the central sourse at a frequency f_0 waves per second. Their separation is the wavelength λ, where $f = v/\lambda$, v being the speed of the waves. A stationary observer will (of course) observe them to reach him with frequency f.



5.1.1 Source is moving



Let S be a sound source emitting a note of true frequency f, and O a stationary observer. Let v be the velocity of sound waves.

If the source S is also stationary, then the f waves emitted by it in an interval of one second will occupy a distance $SO = v$, as shown in figure (a). The first wave emitted at the beginning of the inetrval has arrived at O when the last $\left(f^{th}\right)$ wave is just being emitted from S at the end of the interval. The wavelength is thus $\lambda = \frac{v}{f}$.

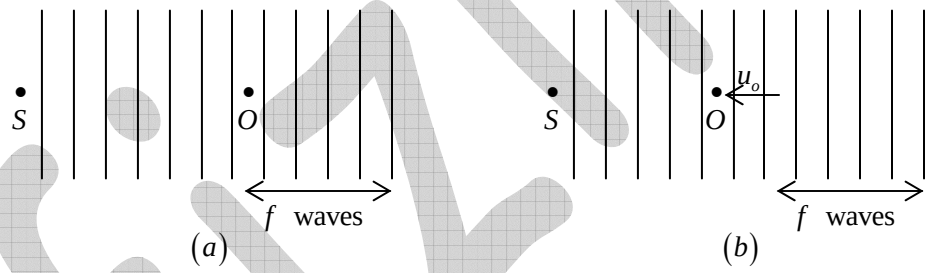
If now S is moving with velocity u_s, the first wave emitted at the beginning of an interval of one second will still have arrived at O, but the fth wave at the end of the interval will be emitted when S has moved through a distance u_s towards O, as shown in figure (b). Thus f waves are crowded into a distance $v-u_s$. Thus wavelength will be decreased to λ' given by

$$\lambda' = \frac{v-u_s}{f}.$$

Thus frequency will be increased to $f' = \frac{v}{\lambda'} = f\left(\frac{v}{v-u_s}\right)$.

Similarly if source is moving away from the observer then $f' = f\left(\frac{v}{v+u_s}\right)$.

5.1.2 Observer is moving



When the observer O is at rest, f waves (of wavelength λ) pass him in one second as shown in figure (a).

If the observer is moving towards the source S with velocity u_o, additional number of waves equal to that contained in a length u_o will pass him per second(no change in wavelength λ). Thus apparent frequency

$$f' = f + \frac{u_o}{\lambda} = f + \frac{u_o}{v/f} = f\left(1+\frac{u_o}{v}\right) \qquad \Rightarrow f' = f\left(\frac{v+u_o}{v}\right)$$

Similarly, if the observer is moving away from the source then $f' = f\left(\frac{v-u_o}{v}\right)$

5.1.3 Both Source and Observer is moving

Case-I

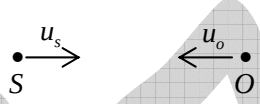


If the observer is at rest, the apparent frequency $f_1 = f\left(\frac{v}{v-u_s}\right)$

If the observer is also moving away from the source $f' = f_1\left(\frac{v-u_o}{v}\right) = f\left(\frac{v-u_o}{v-u_s}\right)$

Effect of Wind: If the wind is blowing with a velocity u_w in the same direction as the sound wave, then $f' = f\left(\frac{v+u_w-u_o}{v+u_w-u_s}\right)$.

Case-II



If the observer is at rest, the apparent frequency $f_1 = f\left(\frac{v}{v-u_s}\right)$

If the observer is also moving away from the source $f' = f_1\left(\frac{v+u_o}{v}\right) = f\left(\frac{v+u_o}{v-u_s}\right)$

Case-III



If the observer is at rest, the apparent frequency $f_1 = f\left(\frac{v}{v+u_s}\right)$

If the observer is also moving away from the source $f' = f_1\left(\frac{v-u_o}{v}\right) = f\left(\frac{v-u_o}{v+u_s}\right)$

Important Note:

In order to solve the numerical problems, the following formula is enough to remember.

When both the source and observer are moving in the direction of sound with velocities u_s and u_o respectively, the apparent frequency is given by

$$f' = f\left(\frac{v-u_o}{v-u_s}\right)$$ where v is velocity of sound and f is actual frequency.

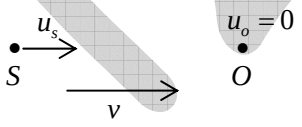
The formulas for all other cases should be obtained by giving proper sign to the velocities u_s and u_o, according to following rule:

(i) The positive direction of sound is taken as the direction from the source towards the observer.

(ii) Velocities measured in the positive direction of sound should be taken as positive, while those measured in opposite direction is negative.

Example: A person is standing near a railway track and a train moving with a speed of $36\ km/hr$ is approaching him. The apparent pitch of the whistle as heard by the person is $700\ Hz$. Calculate the actual frequency of the whistle. The velocity of sound is $350\ m/s$.

Solution:

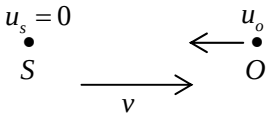


$u_s = +36\ km/hr = +10\ m/s$, $u_o = 0$, $v = 350\ m/s$

$$\Rightarrow 700 = f\left(\frac{350-0}{350-10}\right) \Rightarrow f = 680\ Hz$$

Example: A policeman blows a whistle of frequency $440\ Hz$. A car moves towards the policeman with speed $3\ m/s$. If the velocity of sound is $330\ m/s$, then find the frequency of whistle as heard by a person in the car.

Solution:



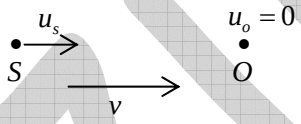
$u_s = 0,\ u_o = -3\ m/s,\ v = 330\ m/s$

$$\Rightarrow f' = 440\left(\frac{330-(-3)}{330-0}\right) \Rightarrow f' = 444\ Hz$$

Example: A locomotive is passing an observer at $72\ km/hr$ and is sounding its whistle whose frequency is $400\ Hz$. Determine the frequencies heard by the observer as the locomotive (a) approaches (b) recedes. Velocity of sound is $332\ m/s$

Solution:

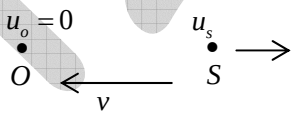
(a)



$u_s = +72\ km/hr = +20\ m/s,\ u_o = 0,\ v = 332\ m/s$

$$\Rightarrow f' = 400\left(\frac{332-0}{332-20}\right) \Rightarrow f' = 426\ Hz$$

(b)

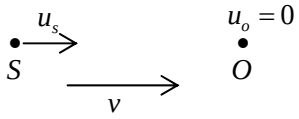


$u_s = -72\ km/hr = -20\ m/s,\ u_o = 0,\ v = 332\ m/s$

$$\Rightarrow f' = 400\left(\frac{332-0}{332-(-20)}\right) \Rightarrow f' = 377\ Hz$$

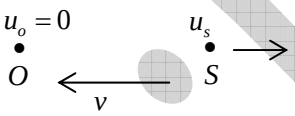
Example: The pitch of the whistle of an engine moving at $72\ km/hr$ appears to drop by $40\ Hz$ as it passes a stationary observer on the railway platform. Find frequency of the whistle. The velocity of sound in air is $350\ m/s$.

Solution:



$u_s = +72\ km/hr = +20\ m/s,\ u_o = 0,\ v = 350\ m/s$

$$\Rightarrow f' = f\left(\frac{350-0}{350-20}\right) = 1.061f$$



$u_s = -72\ km/hr = -20\ m/s,\ u_o = 0,\ v = 350\ m/s$

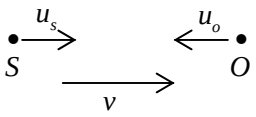
$$\Rightarrow f'' = f\left(\frac{350-0}{350-(-20)}\right) = 0.946f$$

$$\because f' - f'' = 40 \Rightarrow 1.061f - 0.946f = 40 \Rightarrow f = 348\ Hz$$

Example: A train moving with a velocity of $60\ miles/hr$ blows a whistle with frequency $516\ Hz$. Find the note heard by a passenger in another train approaching the first at speed of $30\ miles/hr$. The velocity of sound in air is $750\ miles/hr$.

Solution:

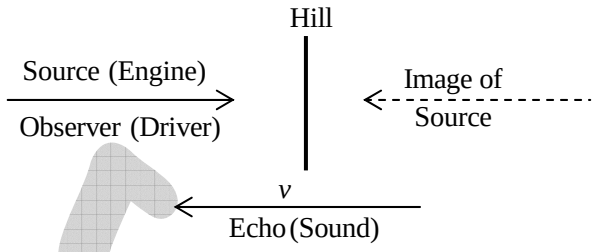
$u_s = +60\ miles/hr,\ u_o = -30\ miles/hr,\ v = 750\ miles/hr$



$$\Rightarrow f' = 516\left(\frac{750-(-30)}{750-60}\right) \Rightarrow f' = 583\ Hz$$

Example: An engine blowing a whistle of frequency 128 Hz moves with a velocity of 72 km/hr towards a hill from which a well defined echo is heard. Calculate the frequency of the echo heard by the driver. The velocity of sound in air is 330 m/s.

Solution:



$u_s = +72\ km/hr = +20\ m/s$,

$u_o = -72\ km/hr = -20\ m/s$, $v = 330\ m/s$

$$\Rightarrow f' = 128\left(\frac{330-(-20)}{330-20}\right) \Rightarrow f' = 144\ Hz$$

Example: A source of sound of frequency 256 Hz is moving rapidly towards wall with a velocity of $5\,m/\sec$. How many beats per second will be heard if sound travels at a speed of $330\,m/\sec$

Solution: If the reflector of sound is stationary the image of the source in the reflecting surface will act as the source of reflected sound. $\therefore f' = f\dfrac{(v+u_0)}{(v-u_S)}$

or $f' = 256\dfrac{(330+5)}{330-5}$ bor $f' = \dfrac{256\times 335}{325} = 264$

$\therefore$ Beat frequency $= f' = f = 264 - 256 = 8\,Hz$.

5.2 Doppler Effect in Light

If a source of light is moving towards an observer, the apparent frequency of the light will be increased (the wavelength will be decreased) and its spectral lines will be shifted towards the blue end of the spectrum. Conversely, if the source is moving away from the observer, the lines will be shifted towards the red end.

There is, however, a difference between the Doppler's effect in sound and that in light. In sound, the Doppler's effect depends upon whether the source is moving or the observer is moving, whereas in case of light it does not.

Suppose the frequency of light emitted from a source is ν. If either source or observer is moving with velocity v such that distance between them is *decreasing*, then the apparent frequency of the source will be given by

$$\nu' = \nu\sqrt{\frac{1+v/c}{1-v/c}} \quad \text{where } c \text{ is speed of light.}$$

Thus frequency appears to be increased (Blue Shifted).

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1-v/c}{1+v/c}} \Rightarrow \frac{\lambda'}{\lambda} = 1-v/c \Rightarrow 1-\frac{\lambda'}{\lambda} = v/c$$

$$\Rightarrow \frac{\Delta\lambda}{\lambda} = \frac{v}{c} \qquad \text{where } \Delta\lambda = \lambda - \lambda'.$$

If either source or observer is moving with velocity v such that distance between them is in*creasing*, then the apparent frequency of the source will be given by

$$\nu' = \nu\sqrt{\frac{1-v/c}{1+v/c}} \quad \text{where } c \text{ is speed of light.}$$

Thus frequency appears to be decreased (Red Shifted).

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1+v/c}{1-v/c}} \Rightarrow \frac{\lambda'}{\lambda} = 1+v/c \Rightarrow \frac{\lambda'}{\lambda} - 1 = v/c$$

$$\Rightarrow \frac{\Delta\lambda}{\lambda} = \frac{v}{c} \qquad \text{where } \Delta\lambda = \lambda' - \lambda.$$

Example: A spectrum line of wavelength $6000\ \overset{0}{A}$ in the spectrum of light from a star is found to be displaced from its normal position toward the red end of the spectrum by an amount of $1\ \overset{0}{A}$. Find the velocity of star if velocity of light is $3\times10^{10}\ cm/\sec$.

Solution:

$$\because \frac{\Delta\lambda}{\lambda}=\frac{v}{c} \qquad \text{where } \Delta\lambda=\lambda'-\lambda.$$

$$\Rightarrow \frac{1}{6000}=\frac{v}{3\times10^{10}} \Rightarrow v=50\times10^{5}\ cm/\sec \Rightarrow v=50\ km/\sec$$

Star moves away from the observer (Red Shifted).

Example: A spectrum line appearing at a wavelength of $4000\ \overset{0}{A}$ in the laboratory is observed at $4005\ \overset{0}{A}$ in the spectrum of a moving galaxy. Find the recession velocity of the galaxy. Velocity of light is $3\times10^{8}\ m/\sec$.

Solution:

$$\because \frac{\Delta\lambda}{\lambda}=\frac{v}{c} \qquad \text{where } \Delta\lambda=\lambda'-\lambda.$$

$$\Rightarrow \frac{5}{4000}=\frac{v}{3\times10^{8}} \Rightarrow v=3.750\times10^{5}\ m/\sec$$

Star moves away from the observer (Red Shifted).

Example: Deduce the Doppler shift if a star emitting light of wavelength $5000\ \overset{0}{A}$ is approaching the observer with velocity of $300\ km/\sec$.

Solution:

$$\because \frac{\Delta\lambda}{\lambda}=\frac{v}{c} \qquad \text{where } \Delta\lambda=\lambda-\lambda'.$$

$$\Rightarrow \Delta\lambda=\frac{300\times10^{5}}{3\times10^{10}}\times5000 \Rightarrow \Delta\lambda=5\ \overset{0}{A}$$

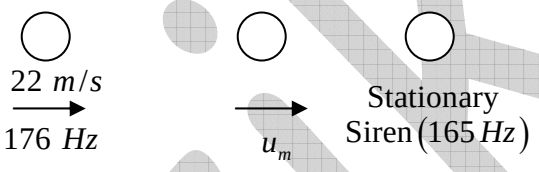
Star moves towards the observer (Blue Shifted).

MCQ (Multiple Choice Questions)

Q1. A siren placed at a railway platform is emitting sound of frequency $5\,kHz$. A passenger sitting in a moving train A records a frequency of $5.5\,kHz$ while the train approaches the siren. During his return journey in a different train B he records a frequency of $6.0\,kHz$ while approaching the same siren. The ratio of the velocity of train B to that train A is

(a) $\frac{242}{252}$ (b) 2 (c) $\frac{5}{6}$ (d) $\frac{11}{6}$

Q2. A police car moving at $22\,m/s$, chases motorcyclist. The police man sounds his horn at $176\,Hz$, while both of them move towards a stationary siren of frequency $165\,Hz$. Then the speed of the motorcycle, if he does not observe any beat is (speed of sound is $330\ m/s$):



(a) $33\,m/s$ (b) $22\,m/s$ (c) zero (d) $11\,m/s$.

Q3. A source of sound of frequency $600\,Hz$ is placed inside water. The speed of sound in water is $1500\,m/s$ and in air it is $300\,m/s$. The frequency of sound recorded by an observer, who is standing in air, is

(a) $200\,Hz$ (b) $3000\,Hz$ (c) $120\,Hz$ (d) $600\,Hz$.

Q4. A source of sound approaches an observer and then recedes from it. Ratio of frequencies of sound as the source approaches and as the source recedes is $6:5$. What is the speed at which the source is moving? (Velocity of sound $= 330\ ms^{-1}$)

(a) $24\ ms^{-1}$ (b) $27\ ms^{-1}$ (c) $30\ ms^{-1}$ (d) $33\ ms^{-1}$

Q5. A fire alarm sounds with a frequency of $480\,Hz$. Two fire engines rush to the site to extinguish the fire from opposite directions. One travels with a speed of $33\ m/s$ and the other with $27.5\ m/s$. If the velocity of sound in air be $330\ m/s$, the differences between the frequencies of the sirens are heard by the drivers of the two fire engines will be:

(a) $8\ Hz$ (b) $12\ Hz$ (c) $16\ Hz$ (d) $24\ Hz$

Numerical Answer Type Questions

Q6. A bus is moving towards a huge wall with a velocity of $5\,ms^{-1}$. The driver sounds a horn of frequency $200\,Hz$. The frequency of the beats heard by a passenger of the bus will be _____Hz (Speed of sound in air $= 342\,ms^{-1}$).

Q7. Two wave of wavelength $50\ cm$ and $50.5\ cm$ produce $6\ beats/s$. What is the velocity of sound ..... $m/\sec$

Q8. A whistle giving out $450\,Hz$ approaches a stationary observer at a speed $33\,m/s$. The frequency heard by the observer in…….. Hz is (Speed of sound $= 330 m/s$)

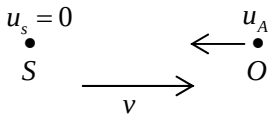
Q9. The ratio of the apparent frequencies of the horn of a car when approaching and receding a stationary observer is $11:9$. If the velocity of sound in air is $300\,ms^{-1}$, the speed of the car is ..... $m/\sec$

Q10. A train moves towards a stationary observer with speed $34\,m/s$. The train sounds a whistle and its frequency registered by the observer is f_1. If the train's speed is reduced to $17\,m/s$, the frequency registered is f_2. If the speed of sound is $340\,m/s$, then the ratio f_1 / f_2 is……

Solution-MCQ (Multiple Choice Questions)

Ans. 1: (b)

Solution:



$$u_s = 0,\ u_o = -u_A,\ \Rightarrow f' = f\left(\frac{v-(-u_A)}{v-0}\right) \Rightarrow 5.5 = 5\left(\frac{v+u_A}{v}\right) \Rightarrow u_A = \frac{v}{10}$$

Similarly $f'' = f\left(\frac{v-(-u_B)}{v-0}\right) \Rightarrow 6 = 5\left(\frac{v+u_B}{v}\right) \Rightarrow u_B = \frac{v}{5}$

Thus $\frac{u_B}{u_A} = \frac{v}{5} \times \frac{10}{v} = 2$

Ans. 2: (b)

Solution: (b): Apparent frequency of police car $f' = f\left(\frac{330-u_m}{330-22}\right) \Rightarrow f' = \left(\frac{330-u_m}{308}\right) \times 176$

Apparent frequency of stationary siren $f'' = f\left(\frac{330+u_m}{330-0}\right) \Rightarrow f'' = \left(\frac{330+u_m}{330}\right) \times 165$

Since no beat is heard by motorcyclist,

$$f' = f'' \quad \therefore \left(\frac{330-u_m}{308}\right) \times 176 = \left(\frac{330+u_m}{330}\right) \times 165 \Rightarrow u_m = 22\, m/\sec .$$

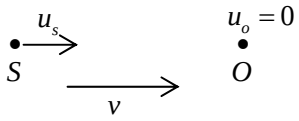
Ans. 3: (d)

Solution: (d): Frequency of sound-wave as of other waves also, does not change during refraction. Frequency of wave depends upon the vibrator which produces the wave. It does not depend on medium in which it is propagated. The wavelength and velocity of the wave undergo change during refraction.

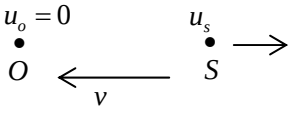
$\therefore$ The observer records the frequency as $600\, Hz$

Ans. 4: (c)

Solution:



$$u_s,\ u_o = 0,\ v = 330\ m/s \Rightarrow f' = f\left(\frac{330-0}{330-u_s}\right) = \frac{330f}{330-u_s}$$



$$-u_s,\ u_o = 0,\ v = 330\ m/s \Rightarrow f'' = f\left(\frac{330-0}{330-(-u_s)}\right) = \frac{330f}{330+u_s}$$

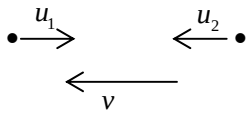
$$\Rightarrow \frac{f'}{f''} == \frac{330+u_s}{330-u_s} = \frac{6}{5} \Rightarrow \frac{6-5}{6+5} = \frac{(330+u_s)-(330-u_s)}{(330+u_s)+(330-u_s)} \Rightarrow \frac{1}{11} = \frac{u_s}{330} \Rightarrow u_s = 30\ m/s$$

Ans. 5: (c)

Solution:

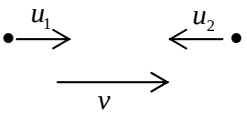
$u_1 = -33\ m/s,\ u_2 = 27\ m/s,\ v = 330\ m/s$

The frequency heard by first engine's drive



$$\Rightarrow f' = 480\left(\frac{330-(-33)}{330-27}\right) = 480\left(\frac{363}{303}\right)$$

$u_1 = 33\ m/s,\ u_2 = -27\ m/s,\ v = 330\ m/s$



Apparent frequency heard by second driver

$$\Rightarrow f'' = 480\left(\frac{330-(-27)}{330-30}\right) = 480\left(\frac{357}{300}\right)$$

Hence, frequency difference $480\left[\frac{363}{303} - \frac{357}{300}\right] = 16\ Hz$

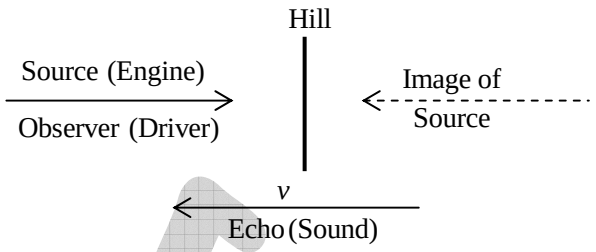
Numerical Answer Type Questions

Ans. 6: 6

Solution:

$u_s = +5\ m/s,\ u_o = -5\ m/s,\ v = 342\ m/s$

The observer and source are moving towards each other. The image of the source serves as source of reflected sound.



$$\Rightarrow f' = 200\left(\frac{342-(-5)}{330-5}\right) \Rightarrow f' = 206\ Hz$$

$\therefore$ Frequency of beats $= 206 - 200\ Hz = 6\ Hz$

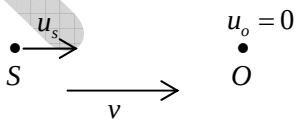
Ans. 7: 303

Solution: $\Rightarrow \text{Beat/s} = \nu_1 - \nu_2 = \frac{v}{\lambda_1} - \frac{v}{\lambda_2} = v\left[\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right]$

$$\Rightarrow \quad 6 = v\left[\frac{1}{0.50} - \frac{1}{0.505}\right] \Rightarrow 6 = 1000\left(\frac{1}{500} - \frac{1}{505}\right)v \Rightarrow v = 303\ m/s$$

Ans. 8: 500

Solution:

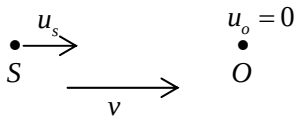


$u_s = +33\ m/s,\ u_o = 0,\ v = 330\ m/s$

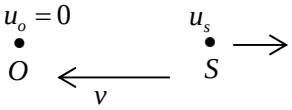
$$\Rightarrow f' = 450\left(\frac{330-0}{330-33}\right) \Rightarrow f' = 500\ Hz$$

Ans. 9: 30

Solution:



$$u_s,\ u_o = 0,\ v = 300\ m/s \Rightarrow f' = f\left(\frac{300-0}{300-u_s}\right) = \frac{300f}{300-u_s}$$



$$-u_s,\ u_o = 0,\ v = 300\ m/s \Rightarrow f'' = f\left(\frac{300-0}{300-(-u_s)}\right) = \frac{330f}{330+u_s}$$

$$\Rightarrow \frac{f'}{f''} == \frac{300+u_s}{300-u_s} = \frac{11}{9} \Rightarrow u_s = 30\ m/s$$

Ans. 10: 1.057

Solution: (d): $f_1 = f\dfrac{v}{(v-u_S)} \Rightarrow f_1 = f\dfrac{340}{(340-34)} = \dfrac{340f}{296}$

$$f_2 = f\frac{v}{(v-u_S)} \Rightarrow f_2 = f\frac{340}{340-17} = \frac{340f}{313}$$

$$\Rightarrow \frac{f_1}{f_2} = \frac{340f}{296} \times \frac{313}{340f} \Rightarrow \frac{f_1}{f_2} = \frac{313}{296} = 1.057$$

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Optics

(IIT-JAM/JEST/TIFR/M.Sc Entrance)

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1. Geometrical Optics

1.1 Fermat's Principle

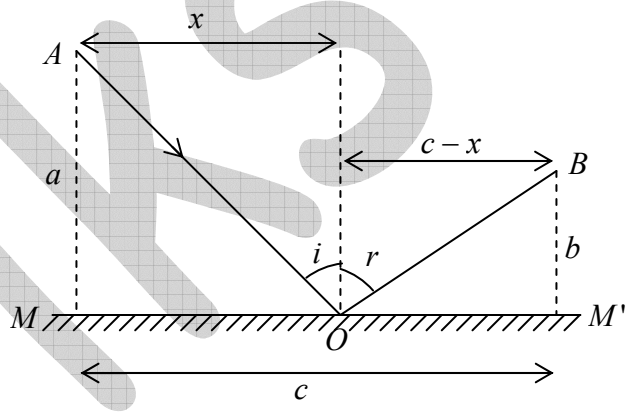
"A ray of light traveling from one point to another by any number of reflections and refractions follows a path for which compared with all other neighboring paths, the time taken is either a minimum or a maximum or stationary." This is known as Fermat's Principle of stationary time or Fermat's Principle of extremum path.

Law of Reflection from Fermat Principle

Let a ray of light travel from point A to B by the reflection at O from the mirror MM'

optical path from A to B

$$L = AO + OB = \sqrt{a^2 + x^2} + \sqrt{(c-x)^2 + b^2}$$



From Fermat's Principle

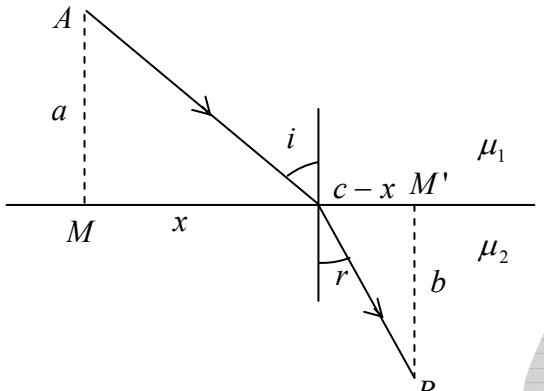
$$\frac{dL}{dx} = 0 \Rightarrow \frac{1}{2}\frac{2}{\sqrt{a^2 + x^2}} - \frac{1}{2}\frac{2x}{\sqrt{(c-x)^2 + b^2}} = 0$$

$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{c-x}{\sqrt{(c-x)^2 + b^2}}$$

$\Rightarrow \sin i = \sin r \Rightarrow i = r$ (This is law of reflection)

In addition, the incident ray, the surface normal and reflected ray all lie in the same plane, called the plane of incidence.

Law of Refraction from Fermat Principle



A light of ray from a point A in medium (1) of refractive index μ_1 be refracted to another point B in medium (2) of refractive index μ_2.

Optical path between A and B

$$L = \mu_1 AO + \mu_2 OB = \mu_1\sqrt{a^2 + x^2} + \mu_2\sqrt{(c-x)^2 + b^2}$$

From Fermat's Principle: $\frac{dL}{dx} = 0 \Rightarrow \mu_1 \frac{x}{\sqrt{a^2 + x^2}} = \mu_2 \frac{(c-x)}{\sqrt{(c-x)^2 + b^2}} \Rightarrow \mu_1 \sin i = \mu_2 \sin r$

$$\Rightarrow \frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1} = {}_1\mu_2 \quad \text{(Snell's Law)}$$

where ${}_1\mu_2$ is the refractive index of the second medium w.r.t. first.

Again the incidence ray, the surface normal and the refracted ray are in the same plane of incidence.

1.2 Refraction and Reflection by spherical surfaces

We will first consider refraction at a spherical surface SPM separating two media of refractive indices n_1 and n_2 as shown in figure (a). Let C represent the center of curvature of the spherical surface. We will consider a point object O emitting rays in all directions. We will use Snell's laws of refraction to determine the image of the point O. We mention that not all rays emanating from O converge to a single point; however, if we consider only those rays which make small angles with the line joining the points O and C, then all rays do converge to a single point. This is known as the paraxial approximation, and according to Fermat's principle all paraxial rays take the same amount of time to travel from O to I.

Now, in terms of the angles defined in figure (a), we have

$$\phi_1 = \beta + \alpha_1 \text{ and } \phi_2 = \beta - \alpha_2$$

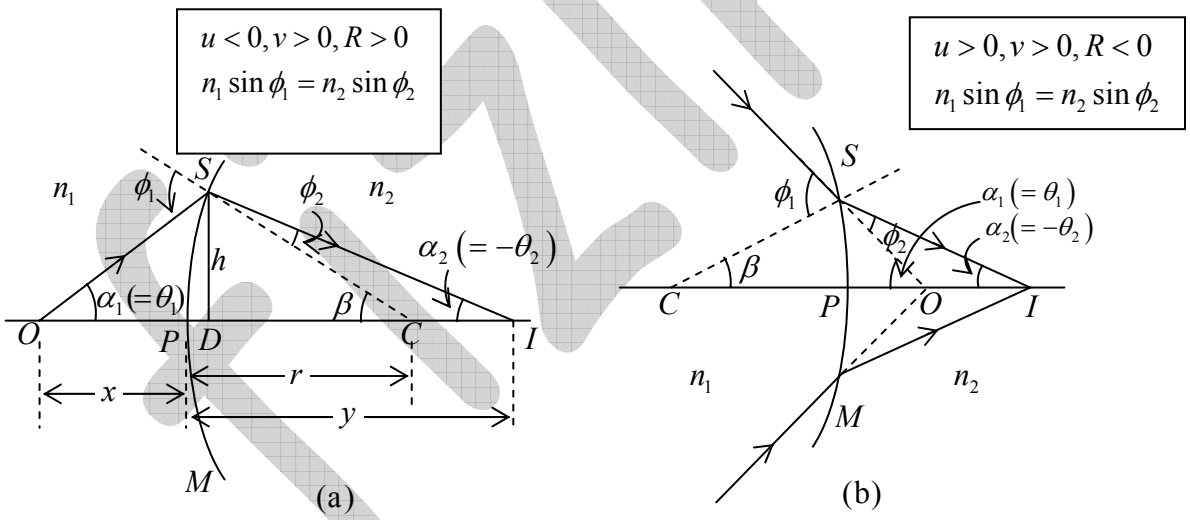


Figure: (a) Paraxial image formation by a spherical refracting surface separating media of refractive indices n_1 and n_2. Point O represents the object point and I the paraxial image point. (b) Corresponds to positive u.

$$\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R}$$

The Sign Convention

1. The rays are always incident from the left on the refracting (or reflecting) surface.

2. All distances to the right of the point P are positive, and distances to the left of the point P are negative.

3. The angle that the ray makes with the axis is positive if the axis has to be rotated in the counterclockwise direction (through the acute angle) to coincide with the ray. Conversely, if the axis has to be rotated in the clockwise direction (through the acute angle) to coincide with the ray, then the slope angle is negative.

4. The angle that a ray makes with the normal to the surface is positive if the normal has to be rotated in the counterclockwise direction (through the acute angle) to coincide with the ray, and conversely.

Reflection by a Single Spherical Surface

We next consider the image of a point object O by a spherical mirror SPM in the paraxial approximation; the point C represents the center of curvature.

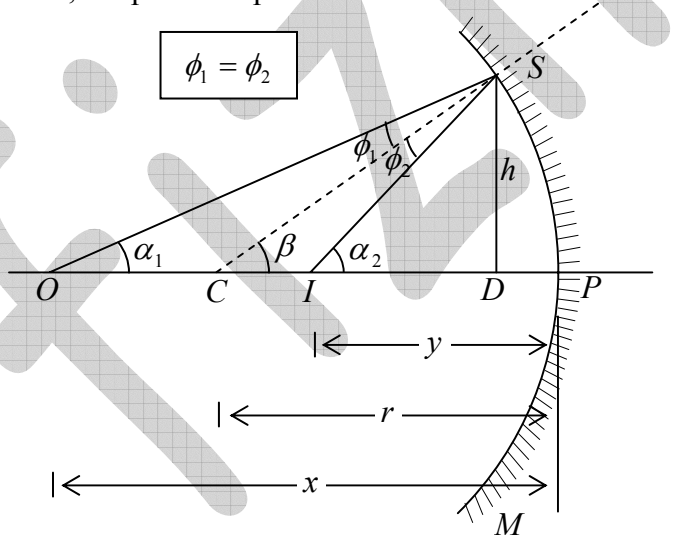


Figure: Paraxial image formation by a spherical reflecting surface SPM.

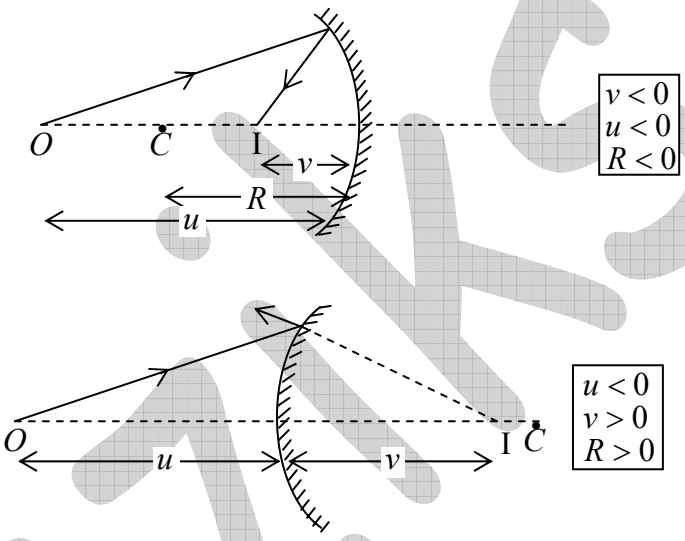
$$\phi_1 = \beta - \alpha_1 \approx \frac{h}{r} - \frac{h}{x} \text{ and } \phi_2 = \alpha_2 - \beta \approx \frac{h}{y} - \frac{h}{r}$$

where the distances x, y, h and r are defined in figure.

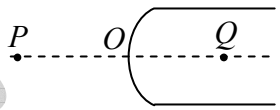
Since $\phi_1 = \phi_2$ (the law of reflection), we get $\frac{1}{x}+\frac{1}{y}=\frac{2}{r}$

If we again use the sign convention that all the distances to the right of P are positive and those to its left negative, then $u=-x, v=-y$ and $R=-r$ thus we obtain the mirror equation $\frac{1}{u}+\frac{1}{v}=\frac{2}{R}$

The above relation is valid for both case give below.



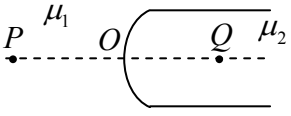
Example: A spherical surface of radius of curvature R separates air (refractive index 1.0) from glass (refractive index 1.5). The centre of curvature is in glass. A point object P placed in air is found to have a real image Q in the glass. The line PQ cuts the surface at a point O and $PO = OQ$. Calculate the distance PQ in term of radius R.



Solution:

By using $\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$ where $\mu_1=1, \mu_2=1.5, u=-OP$ & $v=OQ$

$\frac{1.5}{OQ}-\frac{1}{-OP}=\frac{1.5-1}{R} \Rightarrow \frac{1.5}{OP}+\frac{1}{OP}=\frac{0.5}{R}$ (Since $OP=OQ$)

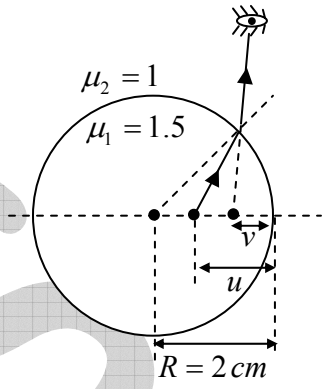


$\Rightarrow \quad \frac{2.5}{OP}=\frac{0.5}{R} \Rightarrow \quad OP=5R$

Example: An air bubble in a glass sphere having $4\ cm$ diameter appears $1\ cm$ from surface nearest to eye when looked along diameter. If ${}_a\mu_g = 1.5$, then calculate the distance of bubble from refracting surface

Solution: By using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$

where $u = ?,\ v = -1\ cm,\ \mu_1 = 1.5,\ \mu_2 = 1$ and $R = -2\ cm$



$$\therefore \frac{1}{-1} - \frac{1.5}{u} = \frac{1-1.5}{(-2)}$$

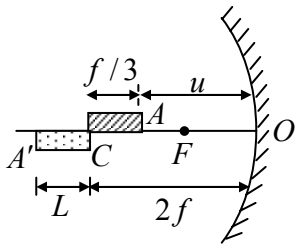
$$\Rightarrow \quad u = \frac{-6}{5} = -1.2\ cm$$

$$u = -1.2\ cm$$

Example: A thin rod of length $\frac{f}{3}$ lies along the axis of a concave mirror of focal length f. One end of its magnified image touches on end of the rod. What is the length of the image?

Solution: If end A of rod acts on object for mirror then its image will be formed at A'. End C remain unchanged and it is only possible if end C is placed at the centre of the concave mirror.

Since Focal length $OF = f \Rightarrow OC = 2f$ and $u = 2f - \frac{f}{3} = \frac{5f}{3}$

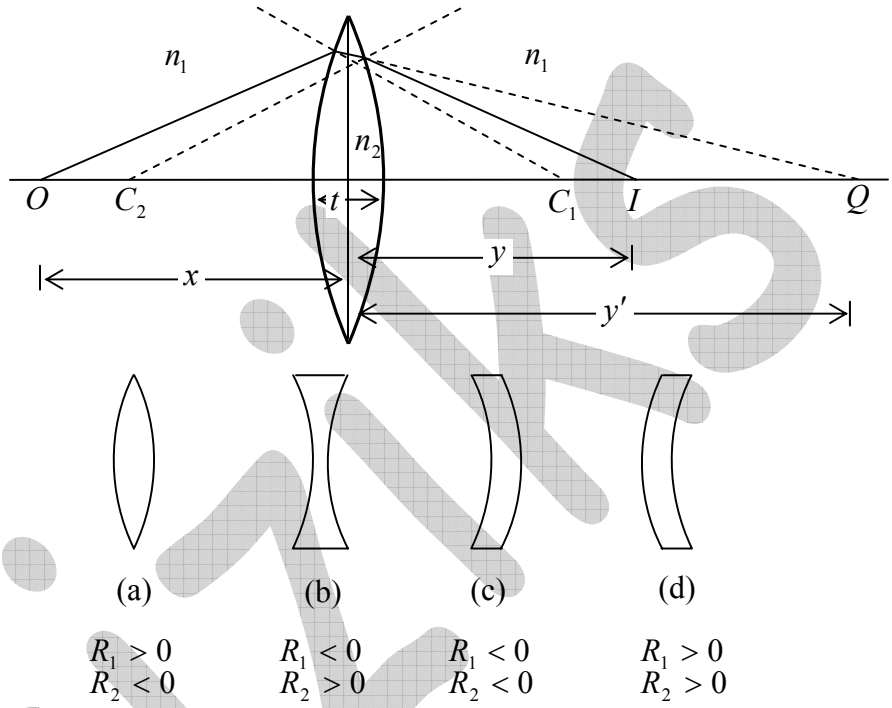


Now by using $\frac{1}{f} = \frac{1}{v} + \frac{1}{u} \Rightarrow -\frac{1}{f} = \frac{1}{v} + \frac{1}{\frac{-5f}{3}} \Rightarrow v = \frac{-5}{2}f$

$\therefore$ Length of the image v is $L = \frac{5f}{2} - 2f = \frac{f}{2} \Rightarrow L = \frac{f}{2}$

1.3 Thin Lens

A medium bounded by two spherical refracting surfaces is referred to as a spherical lens. If the thickness of such a lens t (as shown figure) is very small compared to the object and image distances and to the radii of curvature of the refracting surfaces, then the lens is referred to as a thin spherical lens.



We consider a point object O on the axis of a (thin) lens as shown in figure. The lens is placed in a medium of refractive index n_1 and the refractive index of the material of the lens is n_2. Let R_1 and R_2 be the radii of curvature of the left and right surfaces of the lens, respectively; then R_1 is positive and R_2 is negative. To determine the position of the image, we will consider successive refractions at the two surfaces; the image formed by the first surface is considered the object (which may be real or virtual) for the second surface. Thus, if the second refracting surface had not been there, the image of the point O would have been formed at Q whose position (given by v') is determined from the following equation

$$\frac{n_2}{v'} - \frac{n_1}{u} = \frac{n_2 - n_1}{R_1} \qquad (1)$$

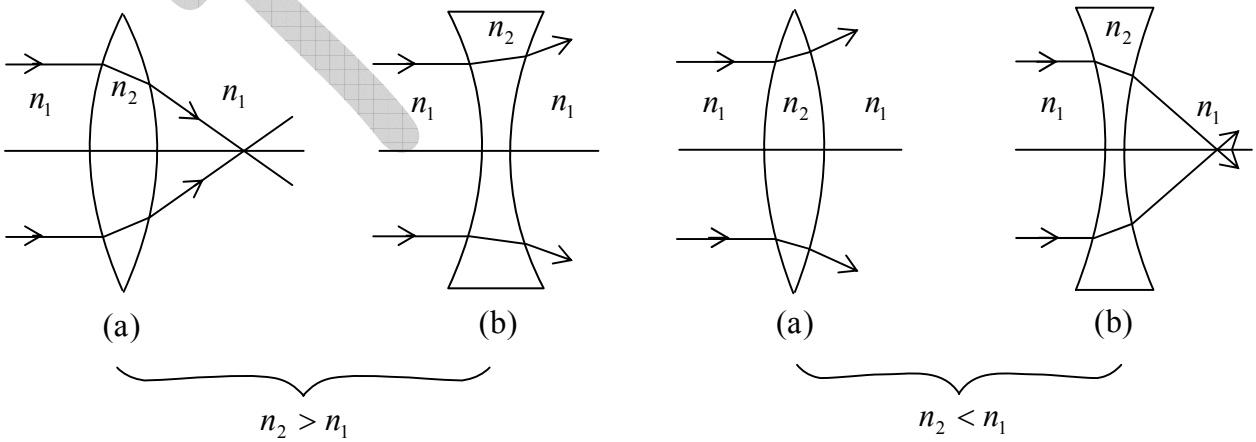
where u is the object distance which is negative for the object point O shown in the figure. Obviously if v' is positive, then the point Q lies to the right of the surface; and if v' is negative, then Q lies to the left of the surface. The point Q now acts as the (virtual) object for the second refracting surface, and the final image is formed at I whose position is determined from the equation $\frac{n_1}{v}-\frac{n_2}{v'}=\frac{n_1-n_2}{R_2}$ (2)

In equation (1) and (2) the distances are measured from the center of the lens P; this is justified because the lens has been assumed to be thin. Adding (1) and (2), we get

$$\frac{1}{f}=\frac{1}{v}-\frac{1}{u}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \text{ where } n=\frac{n_2}{n_1}$$

where f known as the focal length of the lens.

For a lens placed in air (which is usually the case), $n>1$ and if $\frac{1}{R_1}-\frac{1}{R_2}$ is a positive quantity, then the focal length is positive and the lens acts as a converging lens [see figure (a)]. Similarly if $\frac{1}{R_1}-\frac{1}{R_2}$ is a negative quantity, then the lens acts as a diverging lens [see figure (b)]. However, if the double convex lens is placed in a medium whose refractive index is greater than that of the material of the lens, then the focal length becomes negative and the lens acts as a diverging lens [see figure (c)]; similarly for the double concave lens [see Fig. 3(d)].



1.3.1 The Principal Foci and Focal Lengths of a Lens

For a converging lens, the first principal focus is defined as the point (on the axis) such that a ray passing through that point will, after refraction through the lens, emerge parallel to the axis - see ray 1 in figure (a); the point F_1 is the first principal focus. For a diverging lens, the ray which (in the absence of the lens) would have passed through the first principal focus emerges, after refraction by the lens, as a ray parallel to the axis- see ray 1 in figure (b). Point F_1 is the first principal focus, and its distance from the lens (denoted by f_1) is known as the first focal length of the lens. Obviously, f_1 is negative for a converging lens and positive for a diverging lens.

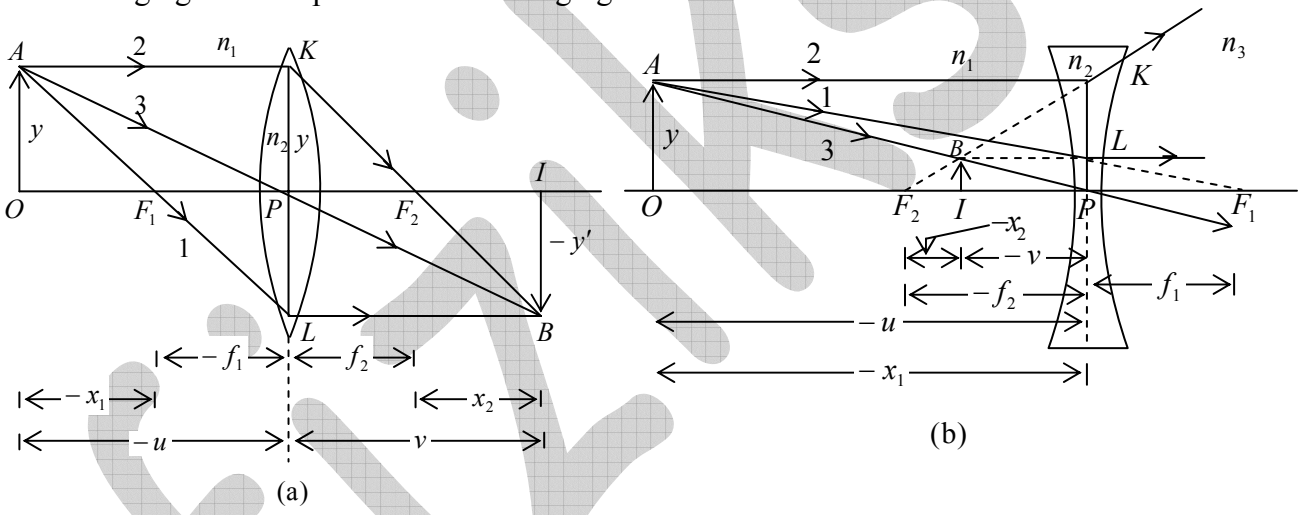


Figure: (a) Paraxial imaging by a converging lens; x_1, f_1 and u are negative quantities and x_2, f_2 and v are positive quantities. (b) Paraxial imaging by a diverging lens; here x_1, x_2, f_2, u and v are negative quantities and f_1 are positive quantities.

We next consider a ray which travels parallel to the axis [see ray 2 in figure (a) and (b)]. For a converging lens the point at which the ray will intersect the axis [shown as F_2 in figure (a)] is known as the second principal focus of the lens.

For a thin lens placed in a medium such that the refractive indices on both sides of the lens are the same ($n_3 = n_1$), the values of f_1 and f_2 can be readily obtained by considering the thin lens formula

$$\frac{1}{f} = -\frac{1}{f_1} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

However, if $n_3 \neq n_1$ then the thin lens formula assumes the following form

$$\frac{n_3}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R_1} + \frac{n_3 - n_2}{R_2}$$

Now, when $v = \infty, u = -f_1$ (ray 1) and we have

$$\frac{1}{f_1} = \frac{1}{n_1}\left(\frac{n_2 - n_1}{R_1} + \frac{n_3 - n_2}{R_2}\right)$$

Similarly, when $u = -\infty, v = f_2$ (ray 2), and we have

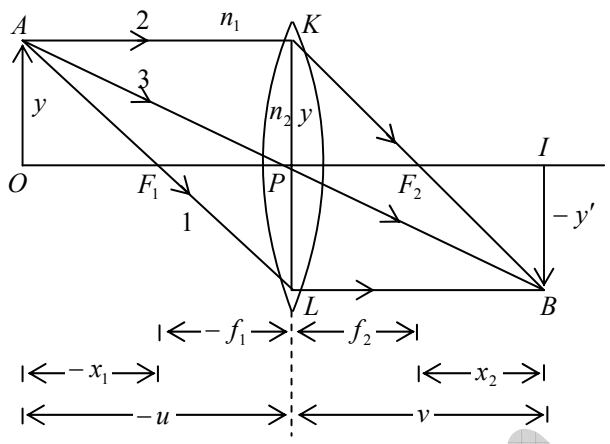
$$\frac{1}{f_2} = \frac{1}{n_1}\left(\frac{n_2 - n_1}{R_1} + \frac{n_3 - n_2}{R_2}\right)$$

Once we know f_1 and f_2 (and therefore the positions of the first and second principal foci), the (paraxial) image can be graphically constructed from the following rules:

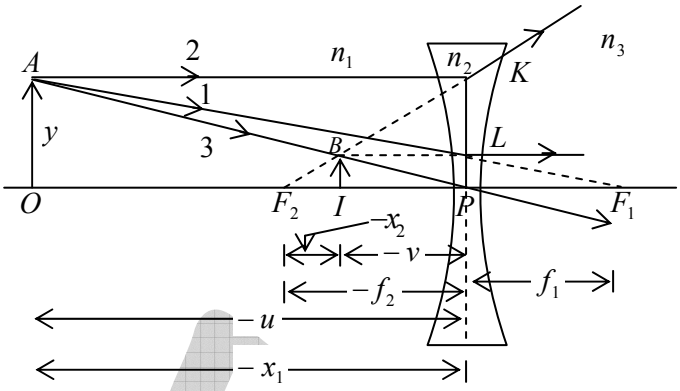
1. A ray passing through the first principal focus will, after refraction, emerge parallel to the axis [see ray 1 in figure (a) and (b)].

2. A ray parallel to the axis will, after refraction, either pass through or appear to come from (depending on the sign of f_2) the second principal focus [see ray 2 in figure (a) and (b)].

1.3.2 The Newton Formula



(a)



(b)

Let x_1 be the distance of object from the first principal focus F_1 (x_1 will be positive if the object point is on the right of F_1 and conversely), and let x_2 be the distance of the image from the second principal focus F_2 as shown in figure (a) and (b). Considering similar triangles in figure (a), we have $\frac{-y'}{y}=\frac{-f_1}{-x_1}$ and $\frac{-y'}{y}=\frac{x_2}{f_2}$

where the vertical distances are positive if measured above the line and negative if below the line.

Thus $$f_1 f_2 = x_1 x_2$$

which is known as the Newtonian lens formula.

It may be noted that for a diverging lens [see figure (b)] $$\frac{y'}{y}=\frac{f_1}{-x_1}=\frac{x_2}{-f_2}$$

When the thin lens has the same medium on the two sides, then

$$x_1 x_2 = -f^2$$

showing that x_1 and x_2 must be of opposite sign. Thus if the object lies on the left of the first principal focus, then the image will lie on the right of the second principal focus, and vice versa.

1.3.3 Magnification

The lateral magnification m is the ratio of the height of the image to that of the object. Considering either figure (a) or figure (b), we readily get

$$m = \frac{y'}{y} = \frac{v}{u} = \frac{f_2 + x_2}{f_1 + x_1} = -\frac{f_1}{x_1} = -\frac{x_2}{f_2}$$

Obviously, if m is positive, the image is erect [as in figure (b)], and conversely if m is negative, the image is inverted [as in figure (a)].

The magnification produced by a single refracting surface is given by

$$m = \frac{y'}{y}$$

Considering triangles AOC and ICB, we get

$$\frac{-y'}{y} = \frac{v - R}{-u + R} = \frac{\frac{v}{R} - 1}{\frac{-u}{R} + 1}$$

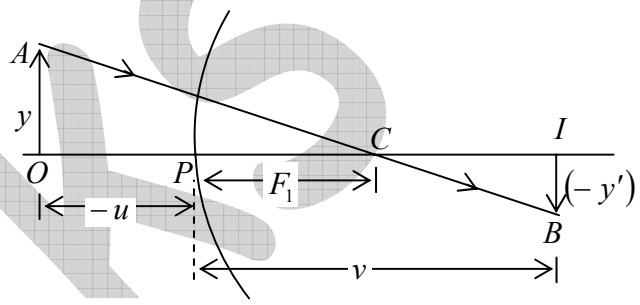


Figure: Imaging of an object of height y by a spherical refracting surface.

Now, equation (5) gives us

$$\frac{n_2}{n_1} - \frac{v}{u} = \frac{n_2 - n_1}{n_1}\frac{v}{R} \quad \text{and} \quad \frac{u}{v} - \frac{n_1}{n_2} = \frac{n_2 - n_1}{n_1}\frac{u}{R}$$

Substance for $\frac{v}{R}$ and $\frac{u}{R}$, we get $m = \frac{y'}{y} = \frac{n_1 v}{n_2 u}$

Thus, if m_1 and m_2 represent the magnifications produced by the two refracting surfaces, then $m_1 = \frac{n_1}{n_2}\frac{v'}{u}$ and $m_2 = \frac{n_2}{n_1}\frac{v}{v'}$

where v' represents the distance of the image formed by the first refracting surface. Thus

$$m = m_1 m_2 = \frac{v}{u}$$

Summary

(1) Lateral (or linear) Transverse Magnification

$$m_y = \frac{\text{Length of the image measured } \perp \text{ to the axis}}{\text{Length of the object measured } \perp \text{ to the axis}} = \frac{\mu_1}{\mu_2}\times\frac{v}{u} \qquad \text{or} \quad m_y = \frac{f_1}{f_2}\times\frac{v}{u}$$

Note: (i) m_y is Positive for erect image and Negative for inverted image

(ii) $m_y = \frac{v}{u}$ for same medium on both side.

(2) Longitudinal Magnification

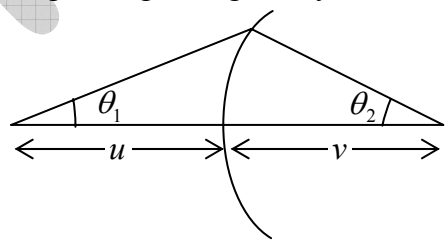
If the object and image have small extension dx_1 and dx_2 respectively along the axis, the ratio dx_2/dx_1 is called the longitudinal magnification.

$$m_x = \frac{dx_2}{dx_1} = \frac{\mu_1}{\mu_2}\times\frac{v^2}{u^2} \Rightarrow m_x = m_y^2\frac{\mu_2}{\mu_1} \Rightarrow m_x \propto m_y^2 \quad \text{or} \quad m_x = -m_y^2\times\frac{f_2}{f_1}$$

(3) Angular magnification

If θ_1 and θ_2 be the angles which the incident and the corresponding emergent rays make with the principal axis then angular magnification

$$m_\theta = \frac{\tan\theta_2}{\tan\theta_1} \Rightarrow m_\theta = \frac{u}{v}$$



I – relation

$$m_x \times m_\theta = \frac{\mu_1}{\mu_2}.\frac{v^2}{u^2}\times\frac{u}{v} = \frac{\mu_1}{\mu_2}.\frac{v}{u} = m_y \Rightarrow m_x \times m_\theta = m_y .$$

i.e. Longitudinal magnification x Angular Magnification = Lateral Magnification.

II – Relation

$$m_x = -m_y^2\times\frac{f_2}{f_1} \Rightarrow m_x \propto m_y^2 \quad \text{but} \quad \frac{f_2}{f_1} = \text{constant}$$

III – Relation

$$m_y \times m_\theta = -\frac{f_1}{f_2}.\frac{v}{u}\times\frac{u}{v} = -\frac{f_1}{f_2} \Rightarrow m_y \times m_\theta = -\frac{f_1}{f_2}$$

Example: A convex lens of focal length $25\,cm$ and made of glass ($_a\mu_g = 1.5$) is immersed in water ($_a\mu_w = \frac{4}{3}$). Calculate the change in the focal length of the lens.

Solution: $\frac{1}{_af_g} = \left(_a\mu_g - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ and $\frac{1}{_wf_g} = \left(_w\mu_g - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

$$\frac{_wf_g}{_af_g} = \frac{_a\mu_g - 1}{_w\mu_g - 1} \Rightarrow {}_wf_g = \frac{_a\mu_g - 1}{\frac{_a\mu_g}{_a\mu_w} - 1} \times {}_af_g \quad \Rightarrow f_w = \frac{\left(_a\mu_g - 1\right)}{\frac{_a\mu_g}{_a\mu_w} - 1} f_a$$

Example: The radius of curvature of the convex face of a Plano- convex lens is $12\,cm$ and its $\mu = 1.5$.

(a) Find the focal length of the lens.

The plane face of the lens is now silvered.

(b) At what distance from the lens will parallel rays incident on the convex surface converge?

(c) Sketch the ray diagram to locate the image, when a point object is placed on the axis $20\,cm$ from the lens.

(d) Calculate the image distance when the object is placed as in (c).

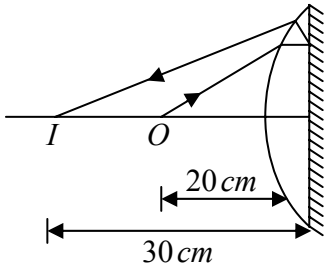
Solution: (a): $\frac{1}{f_I} = (\mu - 1)\frac{1}{R} = (1.5 - 1)\frac{1}{12} = \frac{1}{24}$ or $f_I = 24\,cm$

(b) When a face of the lens is silvered, it behaves as a concave mirror of effective focal length F given by

$$\left|\frac{1}{F}\right| = \frac{2}{f_I} + \frac{2}{f_m} = \frac{2}{f_1} + \frac{1}{\infty} = \frac{2}{f_1} \text{ or } |F| = \frac{f_1}{2} = \frac{24}{2} = 12\,cm$$

This means that the parallel rays will converge at a distance of $12\,cm$ from the silvered surface.

(c) Figure illustrates the formation of the image.



(d) $F=-12\,cm, u=-20\,cm$

$\frac{1}{F}=\frac{1}{v}+\frac{1}{u} \quad \Rightarrow \frac{1}{v}=\frac{1}{F}-\frac{1}{u}=\frac{1}{-12}+\frac{1}{20}$ or $v=-30\,cm$

The negative sign shows that the image is real.

Example: A pin is placed $10\,cm$ in front of a convex lens of focal length $20\,cm$ and made of a material of refractive index 1.5. The surface of the lens farther away from the pin is silvered and has a radius of curvature of $22\,cm$. Determine the position of the final image. Is the image real or virtual?

Solution: $\left|\frac{1}{F}\right|=\frac{1}{f_1}+\frac{1}{f_m}+\frac{1}{f_1}=\frac{2}{f_1}+\frac{1}{f_m}=\frac{2}{20}+\frac{2}{22}$ or $|F|=\frac{110}{21}cm$

Since the convex lens with a silvered surface behaves as a concave mirror of effective focal length F, we have $F=-\frac{110}{21}cm$ and $u=-10\,cm$.

$\because \frac{1}{v}+\frac{1}{u}=\frac{1}{F} \Rightarrow \frac{1}{v}=\frac{21}{110}+\frac{1}{10} \Rightarrow v=-11\,cm$

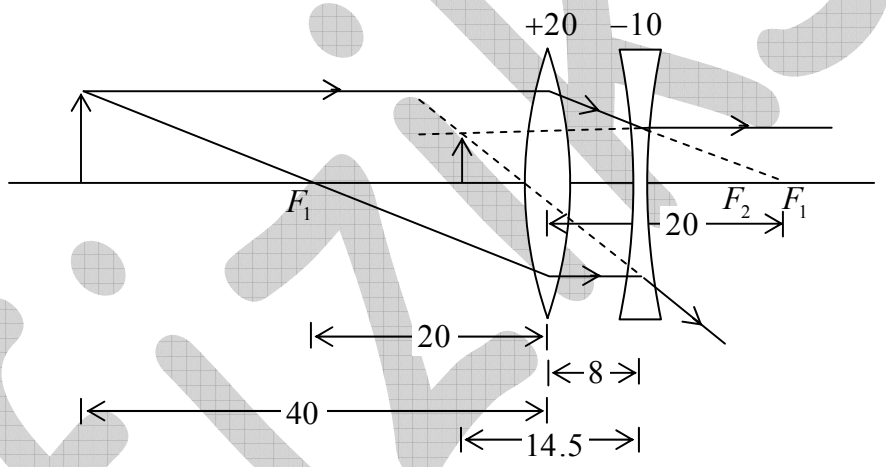
The negative sign shows that the image is in front of the effective mirror and hence is real.

Example: Consider a system of two thin lenses as shown below. The convex lens has a focal length of $+20\ cm$, and the concave lens has a focal length of $-10\ cm$. The two lenses are separated by $8\ cm$. For an object of height $1\ cm$ (at a distance of $40\ cm$ from convex lens), calculate the position and size of the image.

Solution: Let us first calculate the position and size of the image formed by the first lens:

$$u = -40\ cm, \quad f = +20\ cm$$

$\because \frac{1}{v} = \frac{1}{u} + \frac{1}{f} = -\frac{1}{40} + \frac{1}{20} = +\frac{1}{40} \Rightarrow v = +40\ cm$ and $m_1 = -1$; the image is of the same size but inverted. This image acts as a virtual object for the concave lens with $u = +32\ cm$ and $f = +20\ cm$.

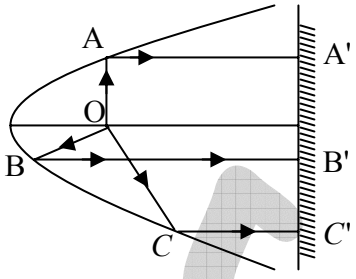


Thick Lens Formula

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2 t}{nR_1R_2}$$

MCQ (Multiple Choice Questions)

Q1. Figure indicates three light rays emerging at the same time, from the focus of a parabolic mirror and reaching the screen which is perpendicular to the axis of parabola. Which ray has the least optical path?



(a) OBB'

(b) OAA'

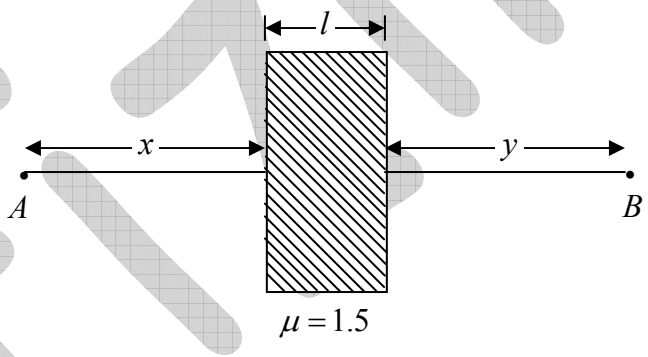
(c) OCC'

(d) All have the same optical path

Q2. A light ray travels through media A and B having refractive indices of 4/3 and 3/2 respectively. If the thickness of the medium A is 4 cm and that of B is 6 cm, then the optical path length in the combined media will be:

(a) 7/3 (b) 12/3 (c) 37/3 (d) 43/3

Q3.



A slab of thickness t and refractive index 1.5 is placed in between points A and B shown in the above figure. The optical path length between A and B is:

(a) $\left(x+\frac{t}{2}+y\right)$ (b) $\left(x+t+y\right)$

(c) $\left(x+\frac{3}{2}t+y\right)$ (d) $\left(x+\frac{5}{2}t+y\right)$

Q4. A convex lens of optical glass $(\mu = 1.42)$ is immersed in a liquid $(\mu = 1.49)$ inside a glass cell. How does the lens-liquid combination act?

(a) As a converging lens (b) As diverging lens

(c) As a plano-parallel lens (d) As a plano-convex lens

Q5. A rod of length 10 *cm* lies along the principal axis of a concave mirror of focal length 10 *cm* in such a way that the end closer to the pole is 20 *cm* away from it. The length of the image of the rod is:

(a) 2 *cm* (b) 5 *cm* (c) 10 *cm* (d) 15 *cm*

Q6. A hollow double concave lens is made of very thin transparent material. It can be filled with air or either of two liquids L_1 or L_2 having refractive indices μ_1 and μ_2 respectively $(\mu_2 > \mu_1 > 1)$. The lens will diverge a parallel beam of light if it is filled with

(a) air and placed in air (b) air and immersed in L_1

(c) L_1 and immersed in L_2 (d) L_2 and immersed in L_1

Q7. The size of the image of an object, which is at infinity, as formed by a convex lens of focal length $30\,cm$ is $2\,cm$. If a concave lens of focal length $20\,cm$ is placed between the convex lens and the image at a distance of $26\,cm$ from the convex lens, calculate the new size of the image.

(a) $1.25\,cm$ (b) $2.5\,cm$ (c) $1.05\,cm$ (d) $2\,cm$.

Q8. A point object is placed at the centre of a glass sphere of radius 6 *cm* and refractive index 1.5. The distance of virtual image from the surface is

(a) $6\,cm$ (b) $4\,cm$ (c) $12\,cm$ (d) $9\,cm$

Q9. A convex lens is in contact with concave lens. The magnitude of the ratio of their focal lengths is $\frac{2}{3}$. Their equivalent focal length is $30\,cm$. Then their individual focal lengths are respectively

(a) 10 *cm*, −15 *cm* (b) −10 *cm*, 15 *cm*

(c) 75 *cm*, −50 *cm* (d) −75 *cm*, 50 *cm*

Q10. An eye specialist prescribes spectacles having combination of convex lens of focal length $40\,cm$ in contact with a concave lens of focal length $25\,cm$. The power of this lens combination, in dioptre, is

(a) +1.5 (b) −1.5 (c) +6.67 (d) −6.67

Q11. Two thin lenses have a combined power of +10 D. When separated by 20 cm, their equivalent power is +6.25 D. Their individual powers, in dioptres, are:

(a) 3.5 and 6.5 (b) 5.0 and 5.0

(c) 7.5 and 2.5 (d) 9.0 and 1.0

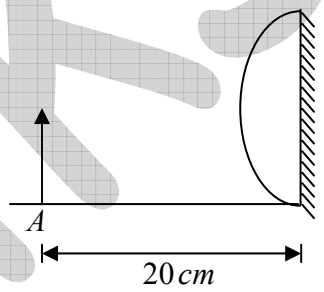
Q12. A point object is placed at a distance of $20\ cm$ from a thin Plano-convex lens of focal length $15\ cm$. A small object is placed at A as shown in the figure. The plane surface is silvered. The image will form at

(a) $60\ cm$ to the left of lens

(b) $60\ cm$ to the right of lens

(c) $12\ cm$ to the left of lens

(d) $12\ cm$ to the right of lens



Q13. An object is placed at $5\ cm$ in front of a concave mirror of radius of curvature $15\ cm$. The position of image and its nature is

(a) $-30\ cm$, Virtual, erect (b) $+20\ cm$, real, inverted

(c) $+15\ cm$, virtual, erect (d) $+15\ cm$, virtual, inverted

Q14. The focal length of a convex lens of refractive index 3/2 is 10 cm in air. Its focal length in water having refractive index 4/3 is:

(a) 40 cm (b) 11.25 cm (c) 9 cm (d) 20 cm

Q15. A convex lens of focal length $0.15\,m$ is made of a material of refractive index $\frac{3}{2}$. When it is placed in a liquid its focal length is increased by $0.225\,m$. The refractive index of liquid is

(a) $\frac{7}{4}$ (b) $\frac{5}{4}$ (c) $\frac{9}{4}$ (d) $\frac{3}{2}$

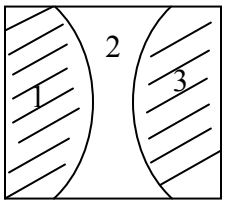
Q16. A diverging meniscus lens of 1.5 refractive index has concave surface of radii 3 and $4\,cm$. The position of the image if an object is placed $12\,cm$ in front of lens is

(a) $7\,cm$ (b) $8\,cm$ (c) $9\,cm$ (d) $10\,cm$

Q17. A spherical surface of radius of curvature R separates air (refractive index 1.0) from glass (refractive index 1.5). The centre of curvature is in the glass. A point object P placed in air is found to have a real image Q in the glass. The line PQ cuts the surface at a point O and $PO = OQ$. The distance PO is equal to

(a) $5R$ (b) $3R$ (c) $2R$ (d) $1.5R$.

Q18. The effective focal length of the lens combination shown in figure is $-60\,cm$. The radii of curvature of the curved surface of the plano-convex lenses are $12\,cm$ each and refractive index of the material of lens is 1.5. The refractive index of the liquid is



(a) 1.33 (b) 1.42 (c) 1.53 (d) 1.60

Q19. Two point light source are $24\,cm$ apart. Where should a convex lens of focal length $9\,cm$ be put in between them from one source so that the image of both the sources are formed at the same place

(a) $6\,cm$ (b) $9\,cm$ (c) $12\,cm$ (d) $15\,cm$

MSQ (Multiple Select Questions)

Q20. When a ray of light of frequency 6×10^{14} Hz travels from water of refractive index 4/3 to glass of refractive index 8/5, its:

(a) Frequency decreases by a factor of $\frac{5}{6}$ (b) speed decreases by a factor of $\frac{5}{6}$

(c) Wavelength increases by a factor of $\frac{5}{6}$ (d) speed increases by a factor of $\frac{5}{6}$

Q21. A converging lens is used to form an image on a screen. When the upper half of the lens is covered by an opaque screen

(a) half the image will disappear

(b) complete image will be formed

(c) intensity of the image wilt increase

(d) intensity of the image will decrease.

Q22. A diminished image of an object is to be obtained on screen 1.0 m from it. This can be achieved by appropriately placing

(a) a concave mirror of suitable focal length

(b) a convex mirror of suitable focal length

(c) a convex lens of focal length less than $0.25\ m$

(d) a concave lens of suitable focal length

Q23. A small candle $2.5cm$ in size is placed at $27\,cm$ in front of a concave mirror of radius of curvature $36\,cm$. The screen is placed at distance L from the mirror to obtain a sharp image. Which of the following statements are correct for the above system

(a) L should be $54\,cm$ in order to obtain sharp image

(b) Magnification of image is 2

(c) Image virtual and erect

(d) If the candle is moved closer to the mirror then the screen will have to be moved away from the mirror in order to obtain images

Q24. Which of the following statements are correct?

(a) a convex mirror always produce a virtual image

(b) a convex mirror produce virtual or real image depending on the location of the object

(c) The convex lens always produce real image dependent of the location of the object

(d) an object placed between the pole and focus of a concave mirror produce a virtual image

Q25. Which of the following statement is correct for concave mirror?

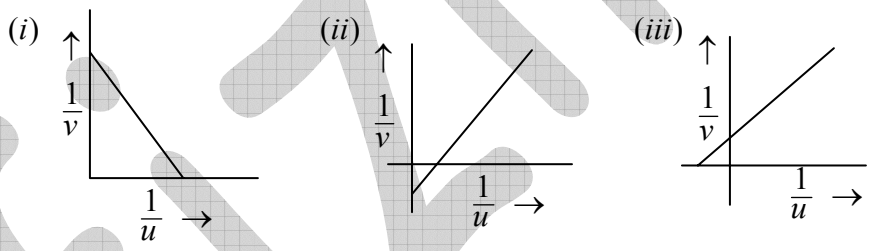
(a) If object is placed at centre of curvature $u = 2f$, the image will be formed $v = 2f$

(b) if object is located at $u = \infty,$ the image formed at $v = f$

(c) If object is located at $u = f$, the image formed at $v = \infty$

(d) If object is placed at $u > 2f$, the image formed at $v > 2f$

Q26. The graphs between $\frac{1}{v}$ and $\frac{1}{u}$ for concave and convex mirror are shown below.



Which of the following statements are correct?

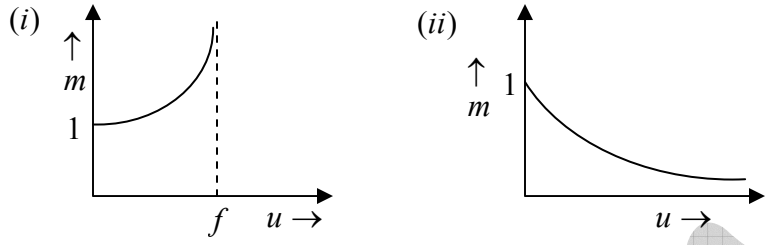
(a) Graph (i) represent for virtual image formed by convex mirror

(b) Graph (ii) represent for virtual image formed by concave mirror

(c) Graph (iii) represent for virtual image formed by convex mirror

(d) Graph (i) represent for real image formed by convex mirror

Q27. The graphs between object distance 'u' and magnification 'm' for concave and convex mirror are given as



Which of the following statement are correct for above graphs?

(a) graph (i) is for virtual image formed by concave mirror

(b) graph (i) is for real image formed by concave mirror

(c) graph (ii) is for virtual image formed by convex mirror

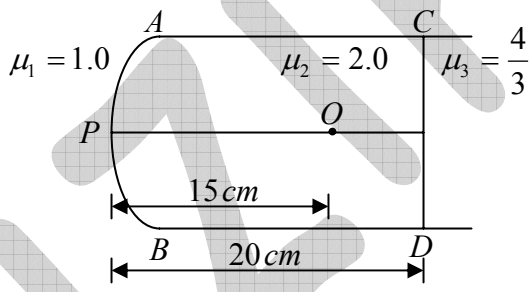
(d) graph (ii) is for real image formed by convex mirror

NAT (Numerical Answer Type)

Q28. A convex lens A of focal length $20\,cm$ and a concave lens B of focal length $5\,cm$ are kept along the same axis with a distance d between them. If a parallel beam of light falling on A leaves B as a parallel beam, then d is equal to _____ cm.

Q29. A thin lens of refractive index 1.5 has a focal length of $15\,cm$ in air. When the lens is placed in a medium of refractive index $\frac{4}{3}$, its focal will become _____ cm.

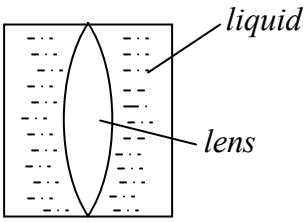
Q30. A slab of a material of refractive index 2 shown in figure has a curved surface APB of radius of curvature $10\,cm$ and a plane surface CD. On the left of APB is air and on the right of CD is water with refractive indices as given in the figure. An object O is placed at a distance of $15\,cm$ from the point P as shown. The distance of the final image of O from P, as viewed from the left is _____cm.



Q31. Two thin lenses, when in contact produce a combination of power $+10$ dioptre. When they are $0.25\ m$ apart, the power reduces to $+6$ dioptre. The focal length of the lenses are _____ m and _____ m.

Q32. Light from a point source in air falls on a spherical glass surface ($\mu = 1.5$ and radius of curvature $R = 20\ cm$). The distance of the light source from the glass surface is $100\ cm$. The position of the image is formed at .....cm

Q33. Shown in figure is a convergent lens placed inside a cell filled with liquid. The lens has focal length $+20\ cm$ when in air and its material has refractive index 1.5. If the liquid has refractive index 1.6, the focal length of the system is .....cm (ignore sign)



Q34. Double-convex lenses are to be manufactured from a glass of refractive index 1.55 with both faces of the same radius of curvature. To keep focal length of $20\ cm$ the radius of curvature required is ……....cm

Q35. A $4.5\ cm$ candle is placed $12\ cm$ away from a convex mirror of focal length $15\ cm$. The height of the image is …….cm

Q36. A beam of light converges at a point P. Now a lens is placed in the path of the convergent beam $12\ cm$ from P. If the lens is a convex lens of focal length $20\ cm$, then beam will converge at point ……cm

Q37. A candle flame $3\ cm$ is placed at distance of $3\ m$ from a wall. In order to obtain an image of flame $9\ cm$ high on wall, how far from wall must a concave mirror be placed ……...cm

Q38. A convex lens of focal length $30\ cm$ and a concave lens of $10\ cm$ focal length one placed so as to have the same axis. If a parallel beam of light falling on convex lens leaves concave lens as a parallel beam, then the distance between the two lenses will be ……..cm

Q39. An object is placed at $10\ cm$ in front of a concave mirror of radius of curvature $15\ cm$. The position of image formed at .....cm

Solution-MCQ (Multiple Choice Questions)

Ans. 1: (d)

Solution: By the property of parabola all rays will have equal path.

Ans. 2: (d)

Solution: Optical path = refractive index ×path in medium

$$= n_A \times d_A + n_B \times d_B = \frac{4}{3}\times 4 + \frac{3}{2}\times 6 = \frac{16}{3} + \frac{18}{2} = \frac{16+27}{3} = \frac{43}{3} cm.$$

Ans. 3: (c)

Solution: Optical path $= \mu t$, where $\mu = \frac{3}{2}$

Total path $= (x + \mu t + y) = x + \frac{3}{2}t + y$

Ans. 4: (b) Solution: $\frac{1}{f} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

$\mu_2 \rightarrow$ refractive index of glass

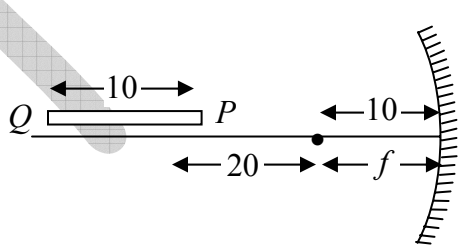
$\mu_1 \rightarrow$ refractive index of the medium

Here, $\mu_2 = 1.42$ and $\mu_1 = 1.49 \Rightarrow \frac{\mu_2}{\mu_1} = \frac{1.42}{1.49} < 1$

It is clear that focal length becomes negative i.e., nature of the lens changes. Hence, convex lens becomes diverging lens.

Ans. 5: (a)

Solution:



Thus point P and Q are at a distance of $(10+20)\ cm$ and $(10+20+10)\ cm$ i.e. $40\ cm$

By mirror formula $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ where $f = -10\ cm$

Image of P, $u=-30\ cm$, $f=-10\ cm$

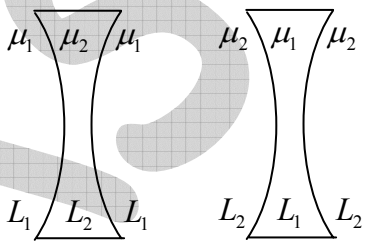
$$\frac{1}{v}+\left(-\frac{1}{30}\right)=\left(-\frac{1}{10}\right) \quad \Rightarrow \frac{1}{v_P}=\frac{1}{30}-\frac{1}{10}=\frac{1-3}{30} \Rightarrow v_P=-15\ cm$$

Image of Q, $u=-40\ cm$, $f=-10\ cm$

$$\frac{1}{v_Q}=\frac{1}{f}-\frac{1}{u}=\left(\frac{1}{-10}\right)-\left(-\frac{1}{40}\right)=\frac{1}{40}-\frac{1}{10}=\frac{1-4}{40}=-\frac{3}{40} \Rightarrow v_Q=-13.3\ cm$$

Thus, length $=-15-(13.3)=-1.7\ cm$

Ans. 6: (d) Solution: $\frac{1}{f}=\left(\frac{\mu_l}{\mu_m}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$



where μ_l = refractive index of material of lens

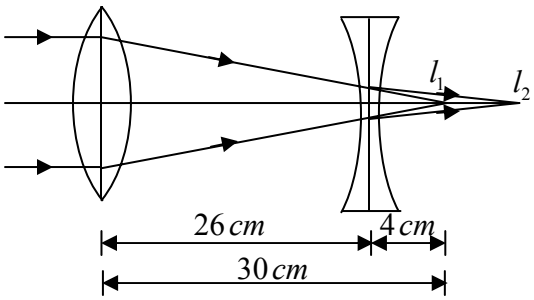
μ_m = refractive index of medium in which lens is dipped

If $\mu_L > \mu_m$, the lens maintains its nature.

A concave lens behaves as a divergent lens and a convex lens behaves as a convergent lens. If $\mu_L < \mu_m$, the nature of lens is reversed. Since $\mu_2 > \mu_1 > 1$, the concave lens will maintain its divergent nature if it is filled with L_2 and immersed in L_1.

Ans. 7: (b)

Solution: Convex lens forms the image at l_1 and is at the second focus of convex lens. Size of $l_1 = 2\,cm$. l_1 acts as virtual object for concave lens. Concave lens forms the image of l_1 at l_2. $\because \frac{1}{v}-\frac{1}{u}=\frac{1}{f}$



For concave lens,

$$\frac{1}{v}-\frac{1}{4}=-\frac{1}{20} \Rightarrow v=5\ cm$$

$$\frac{1}{v}-\frac{1}{4}=-\frac{1}{20}\ \frac{1}{v}=-\frac{1}{20}+\frac{1}{4}=\frac{4}{20}=\frac{1}{5}$$

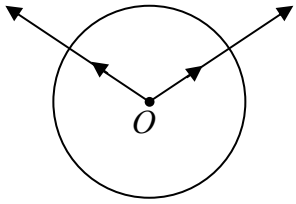
or $v=5\ cm$ (distance of l_2 from concave lens)

$\therefore$ Magnification $= \dfrac{v}{u} = \dfrac{\text{size of image}}{\text{size of object}} = \dfrac{5}{4}$ or $\dfrac{\text{size of image}}{2} = 1.25$

or size of image due to concave lens $= 2.5\ cm$.

Ans. 8: (a)

Solution: A point object is at the centre of a glass sphere of radius $6\,cm$.



The rays from the object fall normally on the surface of the sphere and emerge undeviated. When drawn backwards, they meet at O . The image will be formed at the centre O itself.

$\therefore$ Distance of virtual image from surface $= 6\,cm$

Ans. 9: (a)

Solution: Let focal length of convex lens $= f_1 = f$ (suppose)

$f_2 =$ focal length of concave lens $= \dfrac{-3f}{2}$

Equivalent focal length $= 30\,cm$

$\therefore \dfrac{1}{30} = \dfrac{1}{f} - \dfrac{2}{3f}\left(\because \dfrac{1}{F} = \dfrac{1}{F_1} + \dfrac{1}{F_2}\right)$ or $\dfrac{1}{30} = \dfrac{1}{3f}$

or $f = 10\,cm =$ Focal length of convex lens.

$\therefore$ Focal lengths of concave lens $= \dfrac{-3}{2} \times 10 = -15\,cm$

$\therefore$ Focal lengths are $10\,cm$ (convex lens) and $-15\,cm$ (concave lens).

Ans. 10: (b)

Solution: For convex lens, power $P_1 = \dfrac{1000}{40} = +2.5\,D$

For concave lens, power $P_2 = -\dfrac{1000}{25} = -4.0\,D$

$\therefore$ Power of combination $= P_1 + P_2 = +2.5\,D - 4.0\,D = -1.5\,D$

Ans. 11: (c)

Solution: The equivalent power of two lenses is given as

$P = P_1 + P_2 - d\left(P_1 P_2\right)$ when $d = 0,\ P = P_1 + P_2 \Rightarrow \quad 10 = P_1 + P_2$

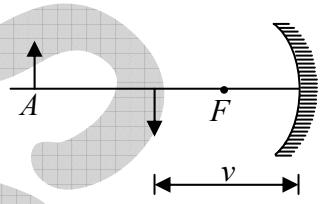
when $d = 0.2\ m,\ P = 6.25\ D$

$\Rightarrow 6.25 = P_1 + P_2 - (0.2) P_1 P_2$. Solving $P_1 = 7.5,\ P_2 = 2.5\ D$

Ans. 12: (c)

Solution: Focal length of equivalent mirror when a lens is silvered at a surface is $\frac{1}{f_1} = \frac{2}{f} + \frac{1}{f_m}$

Here the mirror is a plane $\therefore f_m = \infty$



$$\frac{1}{f_1} = \frac{2}{15} + \frac{1}{\infty} \Rightarrow f_1 = \frac{15}{2} cm$$

Positive focal length means a converging mirror.

For mirror: As per sign convention $u = -20\,cm,\ f_1 = -\frac{15}{2} cm$.

$$\because \frac{1}{f} = \frac{1}{v} + \frac{1}{u} \Rightarrow \frac{1}{-15/2} = \frac{1}{v} + \frac{1}{-20} \Rightarrow \frac{1}{v} = \frac{1}{20} - \frac{2}{15} = \frac{3}{60} - \frac{8}{60} = -\frac{5}{60} \Rightarrow v_1 = -12\,cm$$

Ans. 13: (d)

Solution: By using $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$

The focal length is $f = \frac{-15}{2} = -7.5\ cm$ and $u = -5\ cm$

$$\therefore \frac{1}{v} + \frac{1}{-5} = \frac{1}{-7.5} \Rightarrow \frac{1}{v} = \frac{1}{5} - \frac{1}{7.5} \Rightarrow v = \frac{5 \times 7.5}{7.5 - 5} = +15\,cm$$

This image is formed at $15\ cm$ behind the mirror. If is a virtual image

Magnification $m = \frac{-v}{u} = -\frac{15}{5} = -3$

The image is magnified, virtual and erect

Ans. 14: (a)

Solution: If R_1 and R_2 are radii of spherical surfaces of the convex lens then its focal length in a medium of refractive index μ is given as

$$\frac{1}{f_a}=\left({}_a\mu_g-1\right)\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \quad \text{and} \quad \frac{1}{f_w}=\left({}_w\mu_g-1\right)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

$$\Rightarrow \frac{f_w}{f_a}=\frac{\left({}_a\mu_g-1\right)}{\left({}_w\mu_g-1\right)}=\frac{\left({}_a\mu_g-1\right)}{\left(\frac{{}_a\mu_g}{{}_a\mu_w}-1\right)}$$

$$\Rightarrow \frac{f_w}{f_a}=\frac{(3/2-1)}{\left(\frac{3/2}{4/3}-1\right)}=\frac{1}{2}\times 8=4 \Rightarrow f_w=4f_a=4\times 10\ cm=40\ cm$$

Ans. 15: (b)

Solution: we know $\Rightarrow \frac{f_w}{f_a}=\frac{\left({}_a\mu_g-1\right)}{\left({}_w\mu_g-1\right)}=\frac{\left({}_a\mu_g-1\right)}{\left({}_a\mu_g-{}_a\mu_w\right)}{}_a\mu_w$

$$\Rightarrow \frac{\left({}_a\mu_g-{}_a\mu_w\right)}{{}_a\mu_w}=\left({}_a\mu_g-1\right)\frac{f_a}{f_w} \Rightarrow \frac{{}_a\mu_g}{{}_a\mu_w}=\left({}_a\mu_g-1\right)\frac{f_a}{f_w}+1$$

$$\Rightarrow \frac{3/2}{{}_a\mu_w}=(3/2-1)\frac{0.225}{0.375}+1 \Rightarrow {}_a\mu_w=\frac{5}{4}$$

Ans. 16: (b)

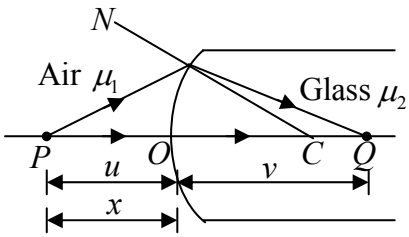
Solution: By using lens formula

$$\frac{1}{f}=(\mu-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \text{ where } R_1=-3\,cm,\ \ R_2=4\ cm,\ \ u=12\ cm$$

$$\therefore \frac{1}{F}=(1.5-1)\left[-\frac{1}{3}+\frac{1}{4}\right]=\frac{1}{v}+\frac{1}{u}=\frac{1}{v}+\frac{1}{12} \Rightarrow v=-8\,cm$$

Ans. 17: (a)

Solution:



When a ray light travels from μ_1 to μ_2 after refraction at a single curved surface,

$$\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$$

As per sign convention, $u=-x, v=+x, R$ is $+ve$,

$$\mu_1=1,\ \mu_2=1.5$$

$$\therefore \frac{1.5}{x}-\frac{1}{-x}=\frac{1.5-1}{R} \quad \text{or,} \frac{1.5}{x}+\frac{1}{x}=\frac{0.5}{R} \text{ or } \frac{2.5}{x}=\frac{0.5}{R} \text{ or } x=5R \ \therefore \quad \text{Distance } PO=5R$$

Ans. 18: (d)

Solution: When lenses are kept in contact $\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}$

$$\frac{1}{f_1}=(\mu-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\Rightarrow\frac{1}{f_1}=(1.5-1)\left(\frac{1}{\infty}-\frac{1}{-12}\right)\Rightarrow\frac{1}{f_1}=\frac{1}{24}$$

$$\frac{1}{f_2}=(\mu-1)\left[\frac{1}{-12}-\frac{1}{12}\right]=-\frac{(\mu-1)}{6}$$

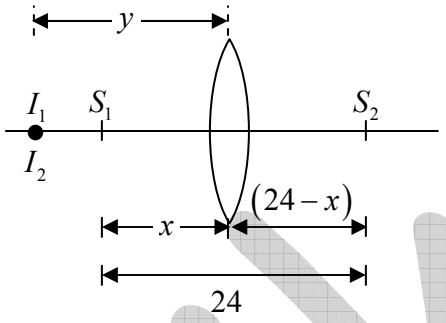
$$\text{and } \frac{1}{f_3}=(1.5-1)\left[\frac{1}{12}-\frac{1}{\infty}\right]=\frac{1}{24}$$

$$-\frac{1}{60}=\frac{1}{24}-\frac{(\mu-1)}{6}+\frac{1}{24} \qquad \because F=-60\,cm$$

$$\frac{(\mu-1)}{6}=\frac{1}{24}+\frac{1}{24}+\frac{1}{60}=\frac{1}{12}+\frac{1}{60}=\frac{1}{10}\Rightarrow(\mu-1)=\frac{6}{10}\Rightarrow\mu=0.6+1\Rightarrow\mu=1.6$$

Ans. 19: (a)

Solution: The given condition will be satisfied only if one source (S_1) placed on one side such that $u < f$ (i.e. it lies under the focus). The other source (S_2) is placed on the other side of the lens such that $u > f$ (i.e. it lies beyond the focus)



If S_1 is the object for lens then $\frac{1}{f}=\frac{1}{-y}-\frac{1}{-x}\Rightarrow\frac{1}{y}=\frac{1}{x}-\frac{1}{f}$

If S_2 is the object for lens then $\frac{1}{f}=\frac{1}{y}-\frac{1}{-(24-x)}\Rightarrow\frac{1}{y}=\frac{1}{f}-\frac{1}{(24-x)}$

$$\frac{1}{x}-\frac{1}{f}=-\frac{1}{f}-\frac{1}{(24-x)}\Rightarrow\frac{1}{x}+\frac{1}{24-x}=\frac{2}{f}=\frac{2}{9}\Rightarrow x^2-24x+108=0$$

On solving we get $x=18\ cm$ and $6\ cm$

The correct answer is option (a)

MSQ (Multiple Select Questions)

Ans. 20: (c), (d)

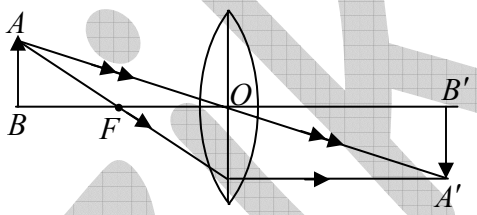
Solution: Frequency is always constant i.e. 6×10^{14} Hz.

$$\text{Speed of light in medium} = \frac{c}{n} \Rightarrow \frac{v_g}{v_w} = \frac{n_w}{n_g} = \frac{4/3}{8/5} = \frac{4}{3}\frac{5}{8} = \frac{5}{6}$$

$$\text{Wavelength in a medium} = \frac{\lambda_0}{n} \Rightarrow \frac{\lambda_g}{\lambda_w} = \frac{n_w}{n_g} = \frac{4/3}{8/5} = \frac{4}{3}\frac{5}{8} = \frac{5}{6}$$

Ans. 21: (b) and (d)

Solution: Image formed will be complete when upper half of lens is blocked. Intensity of the image will decrease as the incident rays from upper half are cut off



Ans. 22: (a) and (c)

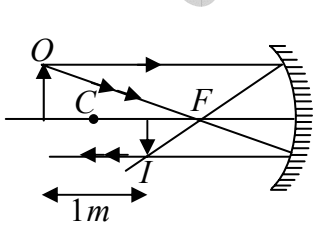
Solution: An image obtained on screen is a real image. Convex mirror and concave lens do not form real image of an object. Options (b) and (d) do not hold good.

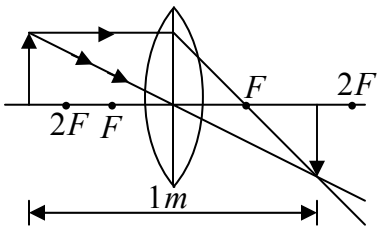
For concave mirror, the object should be beyond C to form a real diminished image between F and C of mirror.

For convex lens, minimum distance between an object and its real image should be $4f$.

$\therefore$ Minimum$(d) = 4f$ or $1m = 4f$, or $f = 0.25m$

Convex lens of focal length less than $0.25m$ can be used when object is placed beyond $(2f)$ and image is formed between $2f$ and f on other side of the lens.





The image formation is illustrated by the figures.

Option (a) and (c) represent the answer.

Ans. 23: (a), (b) and (d)

Solution: The image distance can be obtained $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}\Rightarrow\ \frac{1}{v}=\frac{1}{f}-\frac{1}{u}$

where $u=-27\,cm, f=-\frac{R}{2}=\frac{-36\,cm}{2}=-18\,cm$

$\therefore\frac{1}{v}=\frac{1}{-18}-\frac{1}{-27}=\frac{-3+2}{54}=-\frac{1}{54}\therefore v=-54cm$

Magnification of the image v given as $m=\frac{h'}{h}=-\frac{v}{u}$

$\therefore h'=\frac{-v}{u}\times h=-\left(\frac{-54}{-27}\right)\times 2.5=-5cm\Rightarrow\ m=\frac{h'}{\hbar}=\frac{-5}{2.5}=-2$

The negative sign indicate that the image is inverted and virtual

Ans. 24: (a) and (d)

Solution: (a) For a convex mirror $f>0$ and $u<0$ by using mirror formula

$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}\Rightarrow\ \frac{1}{v}=\frac{1}{f}-\frac{1}{u}\ \Rightarrow\frac{1}{v}=\frac{1}{f}+\frac{1}{u}\Rightarrow v>0$

Thus, the image is formed on the back side of the mirror. Hence a convex mirror always produces a virtual image, regardless of the object distance

Thus (b) and (c) are not correct.

(d) For concave mirror, $f<0$,

when the object is placed on the left side of the mirror, $u<0$

now, It is placed between the focus (f) and the pole

$\therefore f>u>0\Rightarrow\ \frac{1}{f}<\frac{1}{u}<0\Rightarrow\frac{1}{f}-\frac{1}{u}<0$

For image distance v we have the mirror formula $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}\Rightarrow\ \frac{1}{v}=\frac{1}{f}-\frac{1}{u}$

$\therefore\frac{1}{v}<0\ \Rightarrow\ v>0$ .The image v formed on the right side of the mirror.

Hence it is a virtual image.

Ans. 25: (a), (b) and (c)

Solution: (a) By using mirror formula $\frac{1}{u}+\frac{1}{v}=\frac{1}{f} \Rightarrow \frac{1}{v}=\frac{1}{f}-\frac{1}{u}$

For $u=2f$, $\frac{1}{v}=\frac{1}{f}-\frac{1}{2f}=\frac{2f-f}{2f^2}=\frac{1}{2f} \Rightarrow v=2f$

(b) for $u=\infty$, $\frac{1}{\infty}+\frac{1}{v}=\frac{1}{f} \Rightarrow v=f$

(c) For $u=f$, $\frac{1}{v}=\frac{1}{f}-\frac{1}{f}=0 \Rightarrow v=\infty$

(d) For $u>2f$

since $\frac{1}{v}=\frac{1}{f}-\frac{1}{u}=\frac{u-f}{uf} \Rightarrow v=\frac{uf}{u-f}$ (For $u=2f$, $v=2f$ and $u>2f$, $v<2f$)

Ans. 26: (b) and (c)

Solution: For convex mirror $\frac{1}{u}+\frac{1}{v}=\frac{1}{f} \Rightarrow \frac{1}{v}=\frac{1}{f}-\frac{1}{u}$. Since convex mirror always produced virtual image, option (d) is not correct. For virtual image $v>0, u<0$ and $f>0$.

Let $\frac{1}{v}=y$, $\frac{1}{f}=c$ and $\frac{1}{u}=-x \Rightarrow y=c+x$, this is represented by graph (iii).

Thus option (c) is correct.

For Concave mirror $\frac{1}{u}+\frac{1}{v}=\frac{1}{f} \Rightarrow \frac{1}{v}=\frac{1}{f}-\frac{1}{u}$. For virtual image $v>0, u<0$ and $f>0$.

Let $\frac{1}{v}=y$, $\frac{1}{f}=-c$ and $\frac{1}{u}=-x \Rightarrow y=-c+x$, this is represented by graph (ii).

Thus option (b) is correct.

Ans. 27: (a) and (c)

Solution: magnification $m = \dfrac{\text{size of object}}{\text{size of image}} = -\dfrac{v}{u} \qquad \therefore\ m = \dfrac{f}{f-u}$

For concave mirror $f < 0,\ u < 0\ \ \therefore m = \dfrac{-f}{-f+u} = \dfrac{f}{f-u}$

For convex mirror $f > 0, u < 0 \qquad \therefore m = \dfrac{f}{f+u}$

Thus correct options are only (a) & (c)

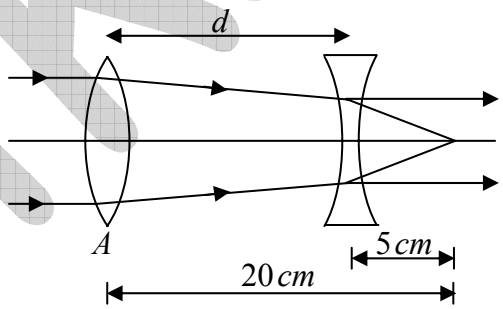
NAT (Numerical Answer Type)

Ans. 28: 15

Solution: The second focus of the convex lens should coincide with the first focus of concave lens for the given image formation.

$d = 20 - 5$

or $d = 15\,cm$.



Solution: $\because \dfrac{1}{f_a} = \left({}_a\mu_g - 1\right)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$ and $\dfrac{1}{f_w} = \left({}_w\mu_g - 1\right)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

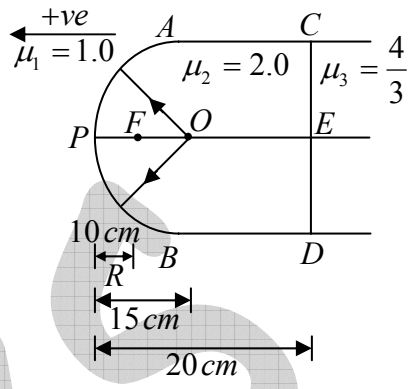
$$\Rightarrow \frac{f_w}{f_a} = \frac{\left({}_a\mu_g - 1\right)}{\left({}_w\mu_g - 1\right)} = \frac{\left({}_a\mu_g - 1\right)}{\left(\dfrac{{}_a\mu_g}{{}_a\mu_w} - 1\right)} \Rightarrow \frac{f_w}{f_a} = \frac{(3/2 - 1)}{\left(\dfrac{3/2}{4/3} - 1\right)} = \frac{1}{2} \times 8 = 4$$

$$\Rightarrow f_w = 4 f_a = 4 \times 15\ cm = 60\ cm$$

Ans. 30: 30 cm

Solution: When light travels from μ_2 medium to μ_1 medium at a single spherical surface, the formula used is

$$\frac{\mu_1}{v}-\frac{\mu_2}{u}=\frac{\mu_1-\mu_2}{R}$$



Direction of light is in positive direction

$$\therefore \frac{1.0}{v}-\frac{2.0}{-15}=\frac{1.0-2.0}{-10}$$

F is centre of curvature of APB

or $\frac{1}{v}=\frac{1}{10}-\frac{2}{15}=\frac{3-4}{30}=\frac{-1}{30}$ or $v=-30\,cm$

$\therefore$ The distance of the final image of O from P, as viewed from the left, is $30\,cm$ to right of P. The image formed will be virtual.

Ans. 31: 0.125 m and 0.5 m

Solution: when the lenses are in contact, $P=P_1+P_2$ or $P_1+P_2=10$

When lenses have d separation,

$$P_1+P_2-dP\,P_2=P \text{ or } P_1+P_2-\frac{P_1P_2}{4}=6 \text{ or } 10-\frac{P_1P_2}{4}=6 \text{ or } P_1P_2=16$$

$$P_1=8D;\ P_2=2D;\quad \therefore f_1=\frac{1}{8}=0.125m,\ f_2=\frac{1}{2}=0.5m$$

Ans. 32: 100 cm

Solution: By using $\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$

where $u=-100\ cm$, $v=?$, $R=+20\,(m)$, $\mu_1=1$, $\mu_2=1.5$ we then have

$$\frac{1.5}{v}-\frac{1}{-100}=\frac{1.5-1}{20}\Rightarrow\frac{1.5}{v}+\frac{1}{100}=\frac{0.5}{20}\Rightarrow v=+100\ cm$$

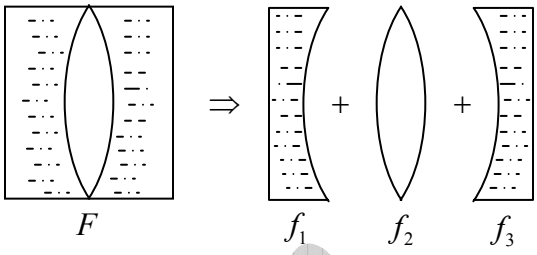
Thus the image is formed at a distance of 100 cm from the glass surface in the direction of incident light.

Ans. 33: 100 *cm*

Solution: $\frac{1}{f_1}=(1.6-1)\left(\frac{1}{\infty}-\frac{1}{20}\right)=\frac{-3}{100}$

$\frac{1}{f_2}=(1.5-1)\left(\frac{1}{20}-\frac{1}{-20}\right)=\frac{1}{20}$

$\frac{1}{f_3}=(1.6-1)\left(\frac{1}{-20}-\frac{1}{\infty}\right)=\frac{-3}{100}$



By using $\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}$ $\Rightarrow$ $\frac{1}{F}=\frac{-3}{100}+\frac{1}{20}\frac{-3}{100}\Rightarrow F=-100\ cm$

Ans. 34: 22 *cm*

Solution: Let $R_1=$ radius of curvature of one face of the lens & $R_2=$ radius of curvature of second face of the length also $R=$ radius of curvature of the double convex lens

$\therefore\quad R_1=R$ and $R_2=-R$

The value of R can be calculated as $\frac{1}{f}=(\mu-1)\left[\frac{1}{R_1}-\frac{1}{R_2}\right]$

$\frac{1}{20}=(1.55-1)\left(\frac{1}{R}+\frac{1}{R}\right)=0.55\times\frac{2}{R}\quad\Rightarrow R=0.55\times2\times20=22\ cm$

Ans. 35: 2.5 *cm*

Solution: The image distance can be obtained as $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}\Rightarrow\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$

where $u=-12\ cm$, $f=15\ cm\ \therefore\quad \frac{1}{v}=\frac{1}{15}+\frac{1}{12}=\frac{4+5}{60}=\frac{9}{60}\Rightarrow v=\frac{60}{9}=6.7\,cm$

The image height can be obtained by the magnification formula

$$m=\frac{h_2}{h_1}=-\frac{v}{u}\Rightarrow h_2=\frac{-v}{u}\times h_1=\frac{-6.7}{-12}\times4.5=2.5\ cm\Rightarrow h_2=2.5\ cm$$

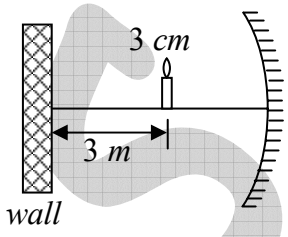
Ans. 36: By using lens formula, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$

where $u=+12\ cm$, $f=20\ cm$ and $v=?$

$$\therefore\quad \frac{1}{v}-\frac{1}{12}=\frac{1}{20}\Rightarrow \frac{1}{v}=\frac{1}{20}+\frac{1}{12}=\frac{3+5}{60}=\frac{8}{60}\Rightarrow v=\frac{60}{8}=7.5\ cm$$

Ans. 37: 450 *cm*

Solution: Let the mirror be placed at a distance x from wall.

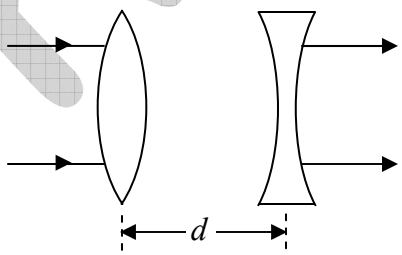


By using $\frac{I}{O}=-\frac{v}{u}\ \Rightarrow\ \frac{-9}{+3}=\frac{-(-x)}{-(x-3)}$

$$\Rightarrow\ x=-4.5\ m=-450\ cm$$

Ans. 38: 20 *cm*

Solution: According to the figure the combination behaves as plane glass plate (i.e. $F=\infty$)



Hence by using

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1f_2}$$

$$\Rightarrow\quad \frac{1}{\infty}=\frac{1}{+30}+\frac{1}{-10}-\frac{d}{(30)(-10)}=\frac{1}{30}-\frac{1}{10}+\frac{d}{300}$$

$$\Rightarrow\quad 0=-\frac{2}{30}+\frac{d}{300}\Rightarrow\frac{d}{300}=\frac{2}{30}\Rightarrow d=20\ cm$$

Ans. 39: 30 *cm*

Solution: By using $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$ where, the focal length $f=\frac{-15}{2}\ cm=-7.5\ cm$, $u=-10\ cm$

$$\therefore\quad \frac{1}{v}+\frac{1}{-10}=\frac{1}{-7.5}\Rightarrow\frac{1}{v}-\frac{1}{10}=\frac{-1}{7.5}\Rightarrow\frac{1}{v}=\frac{1}{10}-\frac{1}{7.5}\Rightarrow v=-30\ cm$$

Thus image is 30 *cm* from the mirror on the same side as the object

2. Interference

Interference of light

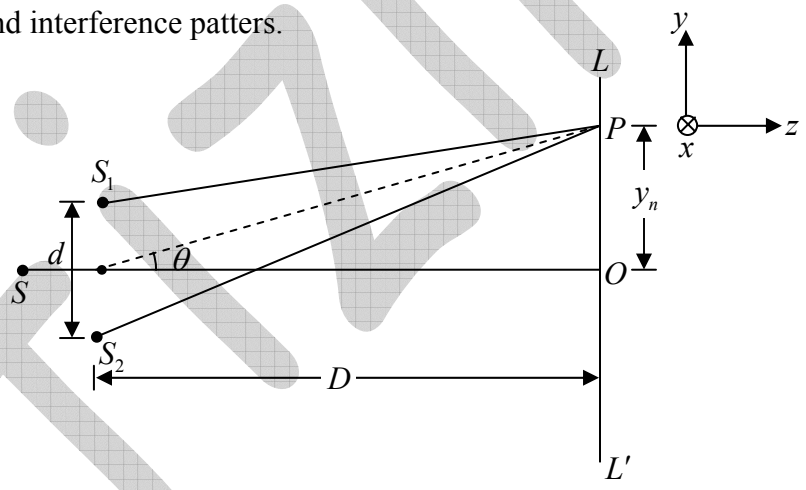
When the two waves of the same frequency travel in approximately the same direction and have a constant phase difference the resultant intensity of the light is not distributed uniformly in space. The non uniform distribution of light intensity due to superposition of two waves is called interference.

At some points the intensity is a maximum and the interference at these points is called constructive interference at some other points, the intensity is a minimum (possibly even zero) and the interference at these points is called destructive interference.

Usually when two lights waves are made to interfere, we get alternate dark and bright bands of a regular or irregular shape.

2.1 Young's double slit Experiment

Purpose of this experiment is to demonstrate the fundamental properties of light diffraction and interference patters.



Let S_1 and S_2 represent the two pinholes of Young's interference experiment. We want to determine the positions of maxima and of minima on line LL' ! which is parallel to the y axis and lies in the plane containing points S, S_1 and S_2. We will show that the interference pattern (around point O) consists of a series of dark and bright lines.

For an arbitrary point P (on line LL') to correspond to a maximum or minima,

$$S_2P - S_1P = n\lambda$$

Now, $(S_2P)^2-(S_1P)^2=\left[D^2+\left(y_n+\frac{d}{2}\right)^2\right]-\left[D^2+\left(y_n-\frac{d}{2}\right)^2\right]=2y_nd$

$$\Rightarrow \qquad S_2P-S_1P=\frac{2y_nd}{S_2P+S_1P}$$

If $y_n,\ d<<D$, then negligible error will be introduced if S_2P+S_1P is replaced by $2D$

$$\therefore \quad S_2P-S_1P=\frac{y_nd}{D} \Rightarrow y_n=\frac{n\lambda D}{d}.$$

Thus, the dark and bright fringes are equally spaced, and the distance between two consecutive dark (or bright) fringes is given by

$$\beta=y_{n+1}-y_n=\frac{\lambda D}{d}$$, which is the expression for the fringe width.

The Intensity Distribution

Let E_1 and E_2 be the electric fields produced at point P by S_1 and S_2, respectively. The electric fields E_1 and E_2 will, in general, have different directions and different magnitudes. However, if the distances S_1P and S_2P are very large in comparison to the distance S_1S_2, the two fields will almost be in the same direction. Thus, we may write

$$\vec{E}_1=\hat{i}E_{01}\cos\left(\frac{2\pi}{\lambda}S_1P-\omega t\right)$$

$$\text{and} \quad \vec{E}_2=\hat{i}E_{02}\cos\left(\frac{2\pi}{\lambda}S_2P-\omega t\right)$$

Where, $\hat{i}$ represents the unit vector along the direction of either of the electric fields. The resultant field is given by

$$E=E_1+E_2=\hat{i}E_{01}\cos\left(\frac{2\pi}{\lambda}S_1P-\omega t\right)+\hat{i}E_{02}\cos\left(\frac{2\pi}{\lambda}S_2P-\omega t\right)$$

The intensity I is proportional to the square of the electric field and is given by $I=KE^2$

$$I=K\left[E_{01}^2\cos^2\left(\frac{2\pi}{\lambda}S_1P-\omega t\right)+E_{02}^2\cos^2\left(\frac{2\pi}{\lambda}S_2P-\omega t\right)+2E_{01}E_{02}\cos\left(\frac{2\pi}{\lambda}S_1P-\omega t\right)\cos\left(\frac{2\pi}{\lambda}S_2P-\omega t\right)\right]$$

$$I = KE_{01}^2 \cos^2\left(\frac{2\pi}{\lambda} S_1P - \omega t\right) + KE_{02}^2 \cos^2\left(\frac{2\pi}{\lambda} S_2P - \omega t\right)$$

$$+KE_{01}E_{02}\left\{\cos\left(\frac{2\pi}{\lambda}(S_2P - S_1P)\right) + \cos\left(2\omega t - \frac{2\pi}{\lambda}(S_2P - S_1P)\right)\right\}$$

where K is a proportionality constant.

The factor $\cos(\omega t - \phi)$ will oscillate between +1 and –1, and its average will be zero, whereas average value of $\cos^2(\omega t - \phi)$ is $1/2$.

Thus the intensity, that a detector will record, will be given by

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta,$$

where $\delta = \frac{2\pi}{\lambda}(S_2P - S_1P)$ represents the phase difference between the displacements reaching point P from S_1 and S_2.

Further $I_1 = \frac{1}{2}KE_{01}^2$ and $I_2 = \frac{1}{2}KE_{02}^2$ represents the intensity produced by source S_1 and from S_2 on the screen.

Condition of Maximum Intensity

The maximum values of $\cos\delta$ is $+1$. Therefore the maximum values of I is given by

$$I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$$

The maximum intensity occurs when, Phase difference: $\delta = 2n\pi;\ \ n = 0,1,2,.....$

or Path difference: $S_2P - S_1P = n\lambda$

Condition of Minimum Intensity

The minimum values of $\cos\delta$ is -1, as a result the minimum values of I is given by

$$I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$$

The minimum intensity occurs when Phase difference: $\delta = (2n+1)\pi;\ \ n = 0,1,2,.....$

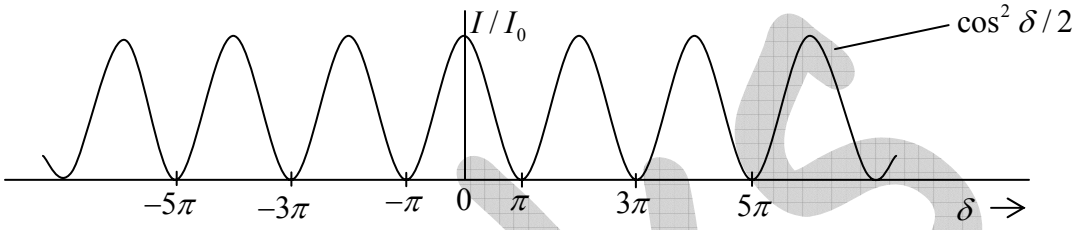
or Path difference: $S_2P - S_1P = \left(n + \frac{1}{2}\right)\lambda$

Intensity when S_1P and S_2P are extremely large

If the distances S_1P and S_2P are extremely large in comparison to d, then $I_1 \cong I_2 = I_0$ and

$$I = 2I_0 + 2I_0 \cos\delta = 4I_0 \cos^2 \delta/2$$

The intensity distribution is shown in below



Conditions for sustained Interference

(1) The two interfering waves should be coherent

(2) The two waves should be of same frequency

(3) If the interfering waves are polarized, they must be in the same state of polarization.

Conditions for observation

(4) The spacing between the light sources (d) should be as small as possible.

(5) The distance (D) of the screen from the two sources should be quite large

Conditions for good contrast

(6) The amplitudes of the interfering waves should be equal or at least very nearly equal.

(7) The two sources must be narrow.

(8) The two sources should give monochromatic or very nearly monochromatic light, or else the path difference should be very small.

Example: Suppose in the double-slit arrangement, $d = 0.150\,mm$, $D = 120\,cm$, $\lambda = 833\,nm$, and $y = 2.00\,cm$.

(a) What is the path difference δ for the rays from the two slits arriving at point P?

(b) Express this path difference in terms of λ

(c) Does point P correspond to maximum, a minimum, or an intermediate condition?

Solutions:

(a) The path difference is given by $\delta = d\sin\theta$. When $D \gg y, \theta$ is small and we can make the approximation $\sin\theta \approx \tan\theta = \frac{y}{D}$. Thus,

$$\delta \approx d\left(\frac{y}{D}\right) = \left(1.50\times10^{-4}\,m\right)\frac{2.00\times10^{-2}\,m}{1.20m} = 2.50\times10^{-6}\,m$$

(b) From the answer in part (a), we have $\frac{\delta}{\lambda} = \frac{2.50\times10^{-6}\,m}{8.33\times10^{-7}\,m} \approx 3.00$ or $\delta = 3.00\lambda$

(c) Since the path difference is an integer multiple of the wavelength the intensity at point P is a maximum.

Example: In Young's double-slit experiment, suppose the separation between the two slits is $d = 0.320\ mm$. If a beam of $500\ nm$ light strike the slits and produces an interference pattern. How many maxima will there be in the angular range $-45.0^{o} < \theta < 45.0^{o}$?

Solution: On the viewing screen, light intensity is a maximum when the two waves interfere constructively. This occurs when

$$d\sin\theta = n\lambda, \qquad n = 0, \pm1, \pm2...$$

where λ is the wavelength of the light. At $\theta = 45.0^{o}$, $d = 3.20\times10^{-4}\,m$ and $\lambda = 500\times10^{-9}\,m$, we get $n = \frac{d\sin\theta}{\lambda} = 452.5$

Thus, there are 452 maxima in the range $0 < \theta < 45.5^{o}$. By symmetry, there are also 452 maxima in the range $-45.0^{o} < \theta < 0$. Including the one for $n = 0$ straight ahead, the total number of maxima is $N = 452 + 452 + 1 = 905$

Example: Let the intensity on the screen at a point P in a double slit interference pattern be 60.0% of the maximum value

(a) What is the minimum phase difference (in radians) between sources?

(b) In (a), what is the corresponding path difference if the wavelength of light is $\lambda = 500\,nm$?

Solution: (a) The intensity $I = I_{\max}\cos^2\left(\frac{\delta}{2}\right)$ where $I_{\max}$ is the maximum light intensity.

Thus, $0.60 = \cos^2\left(\frac{\delta}{2}\right) \Rightarrow \delta = 2\cos^{-1}\left(\sqrt{0.60}\right) = 78.5^{\circ} = 1.37\,rad$

(b) The phase difference δ is related to the path difference Δ and the wavelength λ by

$$\Delta = \frac{\lambda\delta}{2\pi} = \frac{(500\,nm)(1.37\,rad)}{2\pi} = 109\,nm$$

Intensity of Three Slit Interference

Suppose a monochromatic coherent source of light passes through three parallel slits, each separated by a distance d from its neighbor, as shown in following Figure

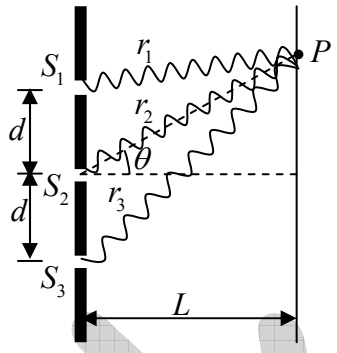


Figure: Three-slit interference

The waves have the same amplitude E_0 and angular frequency ω, but a constant phase difference $\delta = \frac{2\pi}{\lambda} d \sin\theta$.

(a) Show that the intensity is $I = \frac{I_0}{9}\left[1 + 2\cos\left(\frac{2\pi d \sin\theta}{\lambda}\right)\right]^2$ where I_0 is the maximum intensity associated with the primary maxima.

(b) What is the ratio of the intensities of the primary and secondary maxima?

Solution: (a) Let the three waves emerging from the slits be

$$E_1 = E_0 \sin\omega t, \quad E_2 = E_0 \sin(\omega t + \delta), \quad E_3 = E_0 \sin(\omega t + 2\delta)$$

Using the trigonometric identity $\sin\alpha + \sin\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$

The sum of E_1 and E_3 is

$$E_1 + E_3 = E_0\left[\sin\omega t + \sin(\omega t + 2\delta)\right] = 2E_0 \cos\delta \sin(\omega t + \delta)$$

The total electric field at the point P on the screen is

$$E = E_1 + E_2 + E_3 = 2E_0 \cos\delta \sin(\omega t + \delta) + E_0 \sin(\omega t + \delta) = E_0(1 + 2\cos\delta)\sin(\omega t + \delta)$$

where $\delta = \frac{2\pi}{\lambda} d \sin\theta$

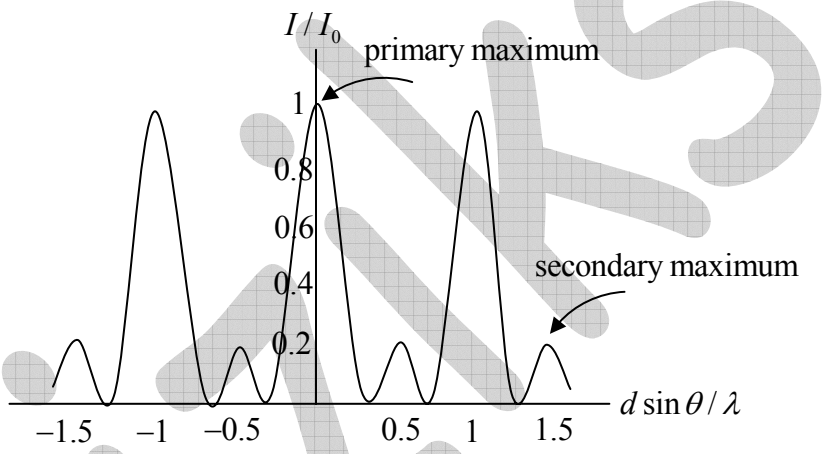
The intensity is proportional to $\langle E^2 \rangle$, thus

$$I \propto E_0^2 (1+2\cos\delta)^2 \langle \sin^2(\omega t+\delta)\rangle = \frac{E_0^2}{2}(1+2\cos\delta)^2 \quad \because \langle \sin^2(\omega t+\delta)\rangle = \frac{1}{2}$$

The maximum intensity I_0 is attained when $\cos\delta = 1$, thus $I_0 = \frac{9E_0^2}{2}$

Thus, $\frac{I}{I_0} = \frac{(1+2\cos\delta)^2}{9} \Rightarrow I = \frac{I_0}{9}(1+2\cos\delta)^2 = \frac{I_0}{9}\left[1+2\cos\left(\frac{2\pi d\sin\theta}{\lambda}\right)\right]^2$

(b) The interference pattern is shown in following figure



Form the figure, we see that the minimum intensity is zero, and occurs when $\cos\delta = -\frac{1}{2}$.

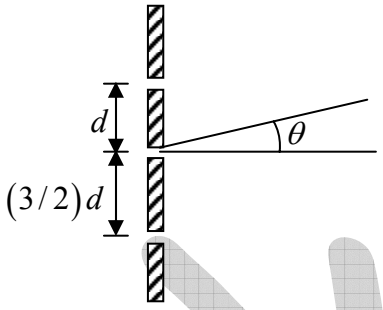
The condition for primary maxima is $\cos\delta = +1$, which gives $\frac{I}{I_0} = 1$.

In addition, there are also secondary maxima which are located at $\cos\delta = -1$. The condition implies

$$\delta = (2m+1)\pi \Rightarrow \frac{d\sin\theta}{\lambda} = \left(m+\frac{1}{2}\right), \qquad m = 0, \pm 1, \pm 2, \ldots$$

The intensity ratio is $\frac{I}{I_0} = \frac{1}{9}$.

Example: Consider the interference pattern from the three slits illustrated (shown in figure). Assume the openings of the individual slits are the same $\left(\le \frac{\lambda}{2}\right)$.At what value of θ is the first principal maximum? (i.e., the wavelets from all three slits are in phase).



Solution: The electric field $E(\theta)$ on the screen is the sum of the fields produced by the slits individually: $E(\theta) = E_1 + E_2 + E_3 = A + Ae^{i\delta} + Ae^{i\frac{5}{2}\delta}$

where $\delta = \frac{2\pi d}{\lambda}\sin\theta$.

Then the total intensity at θ is

$$I(\theta) \sim E(\theta).E^*(\theta) = A^2\left\{3 + 2\left[\cos\delta + \cos\left(\frac{3\delta}{2}\right) + \cos\left(\frac{5\delta}{2}\right)\right]\right\}.$$

For $\theta = 0$, we have $I(0) \sim 9A^2$.

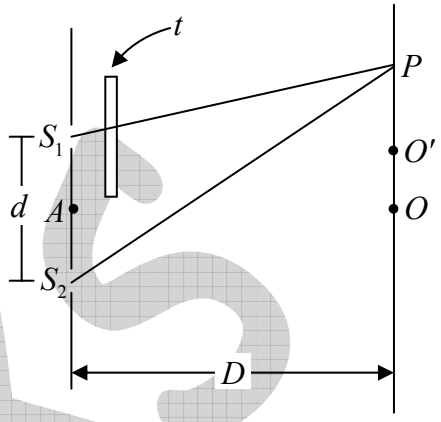
The expression for $I(\theta)$ shows that the first principal maximum occurs at $\delta = 2\pi$, i.e.,

$$\left(\frac{2\pi d}{\lambda}\right)\sin\theta_1 = 2\pi \text{ or } \theta_1 \approx \sin\theta_1 \sim \frac{\lambda}{d}$$

2.1 Displacement of Fringes by the introduction of a thin Lamina Sheet

When a thin transparent plate, say of glass or mica, is introduced in the path of one of the two interfering beams, the entire fringe-pattern is displaced to a point towards the beam in the path of which the plate is introduced.

Let a thin plate of thickness t be introduced in the path of light from S_1. Let μ be the refractive index of the plate for the monochromatic light employed.



Time taken for the journey from S_1 to P

$$\frac{S_1P-t}{v_0}+\frac{t}{v_1}=\frac{S_1P-t}{v_0}+\frac{\mu t}{v_0}=\frac{S_1P+(\mu-1)t}{v_0}$$

v_0 = velocity of light in air &

v_1 = velocity of light in plate

Hence the effective path in air from S_1 to P is $\left[S_1P+(\mu-1)t\right]$

At any point P, effective path difference

$$=S_2P-\left[S_1P+(\mu-1)t\right]=S_2P-S_1P-(\mu-1)t=\frac{d}{D}x_n-(\mu-1)t$$

For bright fringe $\frac{d}{D}x_n-(\mu-1)t=n\lambda \Rightarrow x_n=\frac{D}{d}\left[n\lambda+(\mu-1)t\right]$

In the absence of the plate $t=0$; $x_{n_0}=\frac{D}{d}n\lambda$

Thus the displacement of nth fringe

$$x_0=\frac{D}{d}(\mu-1)t$$

This expression is independent of n so that the displacement of any dark fringe is also

$$\frac{D}{d}(\mu-1)t$$

MCQ (Multiple Choice Questions)

Q1. Two monochromatic light waves of amplitude A and $2A$ interfering at a point have a phase difference of 60°. The intensity at that point will be proportional to:

(a) $3A^2$ (b) $5A^2$ (c) $7A^2$ (d) $9A^2$

Q2. The intensity of light emerging from one slit in a Young's double-slit interference set up is four times that of the other. The ratio of maximum intensity of minimum intensity in the fringe pattern is:

(a) 4 : 1 (b) 2 : 1 (c) 9 : 1 (d) 3 : 1

Q3. Two coherent monochromatic light beams of intensities I and $4I$ are superposed. The maximum and minimum possible intensities in the resulting beam are

(a) $5I$ and I (b) $5I$ and $3I$

(c) $9I$ and I (d) $9I$ and $3I$.

Q4. In a double slit experiment instead of taking slits of equal widths, one slit is made twice as wide as the other. Then, in the interference pattern

(a) the intensities of both the maxima and the minima increase

(b) the intensity of the maxima increases and the minima has zero intensity

(c) the intensity of the maxima decreases and that of the minima increases

(d) the intensity of the maxima decreases and the minima has zero intensity.

Q5. Two beams of light having intensities I and $4I$ interfere to produce a fringe pattern on a screen. The phase difference between the beams is $\frac{\pi}{2}$ at point A and π at point B. Then the difference between the resultant intensities at A and B is

(a) $2I$ (b) $4I$ (c) $5I$ (d) $7I$

Q6. Two light waves having their intensities in the ratio 16:9 interfere to produce interference pattern. What is the ratio of maximum intensity to minimum intensity in this pattern?

(a) $4:3$ (b) $25:7$ (c) $625:49$ (d) $49:1$

Q7. Two coherent light beams are made to interfere and produce interference fringes. If the intensity of one beam is double that of the other, what will be the visibility of fringes?

(a) $1/2$ (b) $\sqrt{2}$ (c) $\sqrt{2}/3$ (d) $2\sqrt{2}/3$

Q8. In Young's double slit experiment, 12 fringes are observed to be formed in a certain segment of the screen when light of wavelength $600\,nm$ is used. If the wavelength of light is changed to $400\,nm$, number of fringes observed in the same segment of the screen is

(a) 12 (b) 18 (c) 24 (d) 30

Q9. The interference pattern is obtained using a yellow light of wavelength $6000\,\overset{0}{\text{A}}$, in which 20 equally spaced fringes occupy 2.0 cm on the screen. On replacing the yellow source by another monochromatic source but making no other changes it is noticed that 30 fringes occupy 2.4 cm on the screen. What is the wavelength of the second source?

(a) 4500 Å (b) 4800 Å (c) 5000 Å (d) 5400 Å

Q10. In a Young's double slit experiment, bichromatic light of wavelengths 400 nm and 560 nm are used. The distance between the slits is 0.1 mm and the distance between the slits and screen is 1 m. The minimum distance between two minima is

(a) 4 mm (b) 5.6 mm (c) 14 mm (d) 28 mm

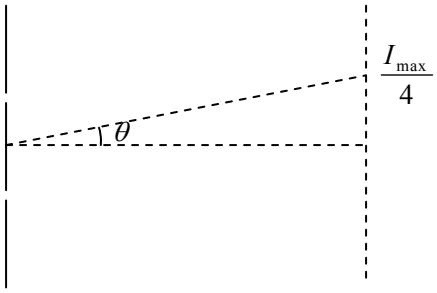
Q11. In a Young's double slit experiment (figure), the angular position (θ) of a bright fringe having intensity one fourth of the maximum intensity is given by

(a) $\sin^{-1}(\lambda/4d)$

(b) $\sin^{-1}(\lambda/3d)$

(c) $\sin^{-1}(\lambda/d)$

(d) $\sin^{-1}(\lambda/d)$



Q12. In the ideal double-slit experiment, when a glass-plate (refractive index 1.5) of thickness t is introduced in the path of one of the interfering beams (wavelength λ), the intensity at the position where the central maximum occurred previously remains unchanged. The minimum thickness of the glass-plate is

(a) 2λ (b) $\frac{2\lambda}{3}$ (c) $\frac{\lambda}{3}$ (d) λ

Q13. A glass plate of refractive index n = 1.5 and thickness 30 mm is placed just below a well-focused microscope. For focusing it again, the microscope should be moved through a distance of:

(a) 45 mm (b) 30 mm (c) 20 mm (d) 15 mm

Q14. In a Young double slit experiment the aperture screen distance is 2 m. The fringe width is 1 mm. Light of 600 nm is used. If a thin plate of glass $(\mu = 1.5)$ of thickness 0.06 mm is placed over one of the slits, then there will be a lateral displacement of the fringes by:

(a) 0 cm (b) 5 cm (c) 10 cm (d) 15 cm

MSQ (Multiple Select Questions)

Q15. In Young's double slit experiment, the interference pattern is found to have an intensity ratio between the bright and dark fringes as 9. This implies that

(a) the intensities at the screen due to the two slits are 5 units and 4 units respectively

(b) the intensities at the screen due to the two slits are 4 units and 1 units respectively.

(c) the amplitude ratio is 3

(d) the amplitude ratio is 2

Q16. White light is used to illuminate the two slits in Young's double slit experiment. The separation between the slits is b and the screen is at a distance $d(>b)$ from the slits. At a point on the screen directly in front of one of the slits, certain wavelengths are missing. Some of these missing wavelengths are

(a) $\lambda = \frac{b^2}{d}$ (b) $\lambda = \frac{2b^2}{d}$ (c) $\lambda = \frac{b^2}{3d}$ (d) $\lambda = \frac{2b}{3d}$

Q17. Which of the following statements is/are correct?

Young's arrangement for observing interference pattern includes

(a) division of wavefront mechanism

(b) division of amplitude mechanism

(c) central bright maxima.

(d) central dark fringe

Q18. Consider the following statements, in case of Young's double-slit experiment

(a) initial slit is necessary if we use an ordinary extended source of light.

(b) initial slit is not needed if we use an ordinary but well collimated beam of light.

(c) initial slit is not needed if we use a spatially coherent source of light.

(d) initial slit is needed if we use a spatially coherent source of light.

Q19. If Young's double-slit experiment is performed under water. Which of the following statement are correct for the interference pattern produced underwater?

(a) Fringe separation will increase (b) Fringe separation will decrease

(c) Intensity of maxima's decreases (d) visibility remains unchanged

Q20. A light source emits visible light of two wavelengths $\lambda_1 = 420\ nm$ and $\lambda_2 = 520\ nm$. The source is used in a double-slit interference experiment in which $D = 100\ cm$ and $d = 0.025\ mm$ which of the following statement are correct for the interference fringes?

(a) Central bright fringes due to λ_1 and λ_2 overlap

(b) First interference maximas of λ_1 and λ_2 are separated by $0.4\ cm$

(c) 42nd bright fringe of λ_2 overlap with 52nd bright fringe of λ_2

(d) on the screen, the bright fringes from λ_1 and λ_2 overlap exactly at $87.4\ cm$

Q21. In a Young's double slit experiment using light of wavelength λ, a thin piece of Plexi glass having refractive index μ covers one of the slits. If the centre point on the screen is a dark spot instead of a bright spot, what is the thickness of the Plexiglass?

(a) $t = \dfrac{\lambda}{\mu - 1}$ (b) $t = \dfrac{\lambda}{2(\mu - 1)}$ (c) $t = \dfrac{3\lambda}{2(\mu - 1)}$ (d) $t = \dfrac{2\lambda}{\mu - 1}$

Q22. A young double slit experiment is performed with white light which of the following statements are correct for interference fringes?

(a) The fringe next to the central maxima will be violet

(b) The central fringe will be white

(c) There will be some dark fringes on the screen

(d) Dark fringes arise due to overlap of fringes produced by different wavelength

Q23. If the source of light used in a Young's double-slit experiment is changed from red to violet, then

(a) Consecutive fringes will be closer

(b) The central bright fringe will be of violet colour and not white

(c) Visibility remains unchanged

(d) The intensity of minima will increase

NAT (Numerical Answer Type Questions)

Q24. In Young's double-slit experiment, the two slits act as coherent sources of equal amplitude A and of wavelength λ. In another experiment with the same set-up the two slits are sources of equal amplitude A and wavelength λ, but are incoherent. The ratio of the intensity of light at the midpoint of the screen, in the first case to that in the second case is _____.

Q25. A Laser beam of wavelength $\lambda = 632.8\ nm$ is incident on two slits $0.20\ mm$ apart. The separation between two consecutive bright interference fringes on screen placed $5.0\ m$ away from the double slits is _____ mm.

Q26. Light of wavelength $460\ nm$ falls on two slits spaced $0.3\ mm$ apart. If the spacing between the first and second dark fringes is to be $4.0\ mm$ than the corresponding distance of screen from slit is _____ m.

Q27. Young's double slit experiment is performed with $589\ nm$ light and a distance of $2\ m$ between the slits and the screen.

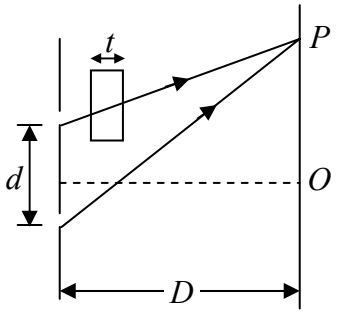
The tenth interference minimum is observed $7.26\ mm$ from the central maximum. The spacing of the slits is _____ mm.

Q28. A Young's interference experiment is performed with monochromatic light. The separation between the slits is $0.5\,mm$ and the interference pattern on screen $3.3\ m$ away shows the first side maximum $3.4\ mm$ from the centre the pattern. The wavelength of the light is _____ nm.

Q29. A Young's double slit experiment with $D = 1.20\,m$ and $d = 0.12\,m$ is performed with light of wavelength $500\ nm$. The phase difference between the two wavefronts arriving at P when $y = 5\ nm$ is _____ rad.

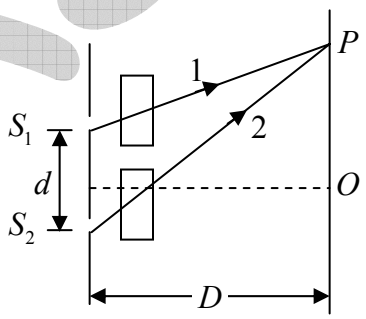
Q30. A Young's double slit experiment is designed with $D = 120\ cm$ and $d = 0.25\,cm$. The slits are illuminated with coherent $600\,nm$ light. Calculate the distance y above the central maximum for which the average intensity on the screen is 75% of the maximum: $y =$ _____ μm.

Q31. In a double-slit interference arrangement one of the slits is covered by a thin mica sheet whose refractive index is 1.58. The slit width is $d = 0.2\,cm$ and $D = 60\,cm$ respectively. Due to the introduction of the mica sheet the central fringe get shifted by $0.4\,cm$. The thickness of the mica sheet is _____ mm.

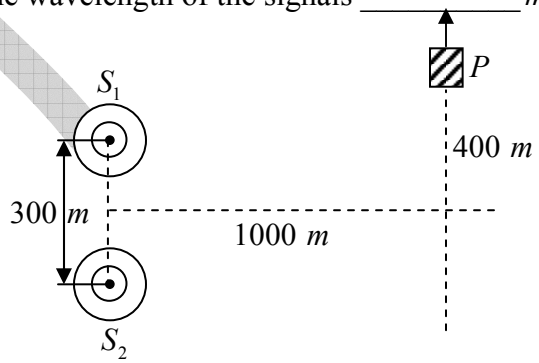


Q32. In Young's double slit experiment one find that by introducing the mica sheet $(\mu = 1.98)$ the central fringe occupies the position that was originally occupied by the 9th bright fringe. If the source of light has wavelength $5893\,\mathring{A}$, then the thickness of the mica sheet is _____ μm.

Q33. In Young's double slit experiment two mica sheets of thickness $0.025\ mm$ each are introduced in the path of both rays emerging from slits S_1 and S_2. The slits width is $d = 50\ cm$. Due to both mica sheet the central fringe get shifted by _____ m.



Q34. Two radio antennas separated by $300\ m$ as shown below, simultaneously broadcast identical signals at the same wavelength. A radio in a car at P traveling due north receives the signals. If the car is at the position of the second maximum what is the wavelength of the signals _____ m.

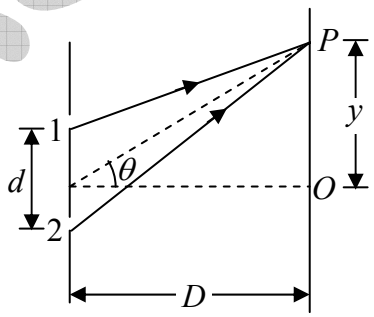


Q35. Two slits are separated by $0.32\ mm$. A beam of $500\ nm$ light strikes the slits, producing an interference pattern. Then the number of maxima's observed in the angular range $-30^{o} < \theta < 30^{o}$ are _____ .

Q36. Two narrow parallel slits separated by $0.25\ mm$ are illuminated by green light $\lambda = 546.1\ nm$. The interference pattern is observed on a screen $1.2\ m$ away from the plane of the slits. The distance from the central maximum to the first bright region on either side of the central maximum is _____ mm.

Q37. A light source emits visible light of two wavelengths $\lambda = 430\ nm$ and $\lambda' = 510\ nm$. The source is used in a double slit interference experiment in which $D = 1.50\ m$ and $d = 0.025\ mm$. The separation distance between the third order bright fringes is _____ mm.

Q38. In a double slit arrangement with $d = 0.15\ mm,\ D = 140\ cm, \lambda = 643\ nm$ and $y = 1.8\ cm$ the path difference δ far the rays from the two slits arriving at P is _____ μm



Solution-MCQ (Multiple Choice Questions)

Ans. 1: (c)

Solution: The resultant amplitude of light waves of amplitude a_1 and a_2 at point when phase difference is δ is

$R^2 = a_1^2 + a_1^2 + 2a_1a_2\cos\delta$ where $a_1 = A,\ a_2 = 2A,$ and $\delta = 60^0$

$$\Rightarrow R^2 = A^2 + 4A^2 + 2A(2A)\cos 60^0 = 5A^2 + 4A\cos 60^0 = 5A^2 + 4A^2\left(\frac{1}{2}\right)$$

$$R^2 = 7A^2 \Rightarrow I \propto 7A^2$$

Ans. 2: (c)

Solution: If a_1 and a_2 are amplitudes of two rays, then the ratio of maximum intensity to the minimum intensity in the fringe pattern is given as

$$\frac{I_{\max}}{I_{\min}} = \frac{(a_1 + a_2)^2}{(a_1 - a_2)^2}$$

According to the question $\frac{I_1}{I_2} = 4 \Rightarrow \frac{a_1^2}{a_2^2} = 4 \Rightarrow \frac{a_1}{a_2} = 2$

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{1 + \frac{a_1}{a_2}}{\frac{a_1}{a_2} - 1}\right)^2 = \left(\frac{1+2}{2-1}\right)^2 = \left(\frac{3}{1}\right)^2 = 9:1$$

Ans. 3: (c)

Solution: $I_{\max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 = \left(\sqrt{4I} + \sqrt{I}\right)^2 = 9I$

$$I_{\min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2 = \left(\sqrt{4I} - \sqrt{I}\right)^2 = I$$

Ans. 4: (a)

Solution: When slits are of equal width,

$I_{\max} \propto \left(\sqrt{I} + \sqrt{I}\right)^2 \propto 4I = 4KI$ (i) and $I_{\min} \propto \left(\sqrt{I} - \sqrt{I}\right)^2 \propto \text{zero} = \text{zero}$ (ii)

In second case, $\frac{I_1}{I_2} = \frac{W_1}{W_2} = \frac{W}{2W} = \frac{1}{2} \Rightarrow I_2 = 2I_1$

$\because I'_{max} = K\left(\sqrt{I_1}+\sqrt{I_2}\right)^2$

$I'_{max} = K\left(\sqrt{I_1}+\sqrt{2I_1}\right)^2 = KI_1\left(1+\sqrt{2}\right)^2 = KI_1\times(2.414)^2 = 5.8\,KI_1$ (iii)

$I'_{min} = K\left(\sqrt{I_1}-\sqrt{I_2}\right)^2 = KI_1\left(1-\sqrt{2}\right)^2 = KI_1(-0.414)^2 = 0.17\,KI_1$ (iv)

From (iii) and (i) $I'_{max} > I_{max}$ and from (iv) and (ii) $I'_{min} > I_{min}$

Ans. 5: (b)

Solution: $I(\phi) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$

At A when $\phi = \frac{\pi}{2}$, $I_A = I + 4I = 5I$ $\quad\because I_1 = I,\ I_2 = 4I$

Again at B when $\phi = \pi$, $I_B = I + 4I + 2\sqrt{I\times 4I}\cos\pi \Rightarrow I_B = 1$

$\therefore I_A - I_B = 5I - I \Rightarrow I_A - I_B = 4I$

Ans. 6: (d)

Solution: If I_1 and I_2 are intensities of two waves then

$$\frac{I_{max}}{I_{min}} = \left[\frac{\sqrt{I_1}+\sqrt{I_2}}{\sqrt{I_1}-\sqrt{I_2}}\right]^2 = \left[\frac{\sqrt{I_1/I_2}+1}{\sqrt{I_1/I_2}-1}\right]^2$$

since, $\frac{I_1}{I_2} = \frac{16}{9} = \left(\frac{\sqrt{16/9}+1}{\sqrt{16/9}-1}\right)^2 = \left(\frac{4/3+1}{4/3-1}\right)^2 = \left(\frac{7/3}{1/3}\right)^2 = \left(\frac{7}{1}\right)^2 = 49:1$

Ans. 7: (d)

Solution: $\text{visibility} = \frac{I_{max}-I_{min}}{I_{max}+I_{min}} = \frac{(a_1+a_2)^2-(a_1-a_2)^2}{(a_1+a_2)^2+(a_1-a_2)^2} = \frac{4a_1a_2}{2\left[a_1^2+a_2^2\right]} = \frac{2\sqrt{I_1I_2}}{(I_1+I_2)}$ $\quad\left[\because\ I_1 \propto a_1^2\right]$

Here, $I_1 = I,\ I_2 = 2I$

So, $\text{visibility} = \frac{2\sqrt{2}}{(2+1)} = \frac{2\sqrt{2}}{3}$

Ans. 8: (b)

Solution: Fringewidth $\beta = \frac{\lambda D}{d}$

Let y is the segment on screen which contain 12 fringes of wavelength λ_1

$\therefore \qquad y = 12\beta = 12\left(\frac{\lambda_1 D}{d}\right)$

Let N fringes of λ_2 occupy the same segment.

$\therefore \qquad N\left(\frac{\lambda_2 D}{d}\right) = 12\left(\frac{\lambda_1 D}{d}\right)$ or $N = \frac{12\times 600\times 10^{-9}}{400\times 10^{-9}}$ or $N = 18$

Ans. 9: (b)

Solution: If D is separation between sources and screen the fringe width is given as

$$\beta = \frac{\lambda D}{d} \qquad \Rightarrow \frac{\beta}{\lambda} = \frac{D}{d}$$

where d = separation between virtual sources. If d and D are fixed, then

$$\frac{\beta_1}{\lambda_1} = \frac{\beta_2}{\lambda_2}$$

In first case 20 fringes occupy 2 cm $\Rightarrow \beta_1 = \frac{20}{20} cm = 0.1\ cm$

In second case 30 fringes occupy $2.4\,cm \Rightarrow \beta_2 = \frac{24}{30} cm = 0.08\ cm$

so, by equation (i), we get $\lambda_2 = \lambda_1 \frac{\beta_2}{\beta_1} = 6000\times\frac{0.08}{0.10} = 4800\,\text{Å}$

Ans. 10: (d)

Solution: The condition for minima (dark fringes) is

$$y_n \frac{d}{D} = (2n+1)\frac{\lambda}{2}$$

For two wavelengths λ_1 and λ_2, we get

$$y_n \frac{d}{D} = (2n+1)\frac{\lambda_1}{2}$$

$$y_m \frac{d}{D} = (2m+1)\frac{\lambda_2}{2}$$

For $y_n = y_m$, we get

$$(2n+1)\lambda_1 = (2m+1)\lambda_2$$

i.e. $$\frac{2n+1}{2m+1} = \frac{\lambda_2}{\lambda_1} = \frac{560nm}{400nm} = \frac{7}{5}$$

This gives $$n = \frac{7m+1}{5}$$

The possible values are $n_1 = 3$, $m_1 = 2$; $n_2 = 10$, $m_2 = 7$ and so on.

Area where the minima of both wavelengths coincides is the area of total darkness. The width of this area is

$$\Delta y = \frac{\lambda_1 D}{d}\left(n_2 + \frac{1}{2}\right) - \frac{\lambda_1 D}{d}\left(n_1 + \frac{1}{2}\right)$$

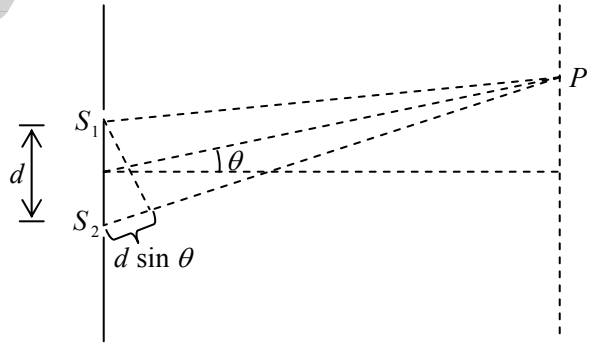
$$= \frac{\lambda_1 D}{d}(n_2 - n_1) = \frac{(400\times10^{-9}m)(1m)}{(10^{-4}m)}(10-3)$$

$$= 2800\times10^{-5} = 28\,m$$

Ans. 11: (b)

Solution: Since $I = I_{\max}\cos^2\left(\frac{\delta}{2}\right)$

$$\Rightarrow \frac{I}{I_{\max}} = \frac{1}{4} = \cos^2\left(\frac{\delta}{2}\right) \Rightarrow \delta = \frac{2\pi}{3}$$



The phase difference of the waves from the two slits S_1 and S_2 is given by $\delta = \left(\frac{2\pi}{\lambda}\right)d\sin\theta$

Hence, $\frac{2\pi}{3} = \left(\frac{2\pi}{\lambda}\right)d\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{\lambda}{3d}\right)$

Ans. 12: (a)

Solution: Path difference due to slab $(\mu-1)t$. For maximum intensity path difference should be equal $n\lambda$. Thus $(\mu-1)t = n\lambda$

For minimum thickness t of plate, n should be minimum i.e. $n=1$

$$\text{or } t=\frac{\lambda}{\mu-1} \text{ or } t=\frac{\lambda}{1.5-1} \quad \text{or } t=2\lambda$$

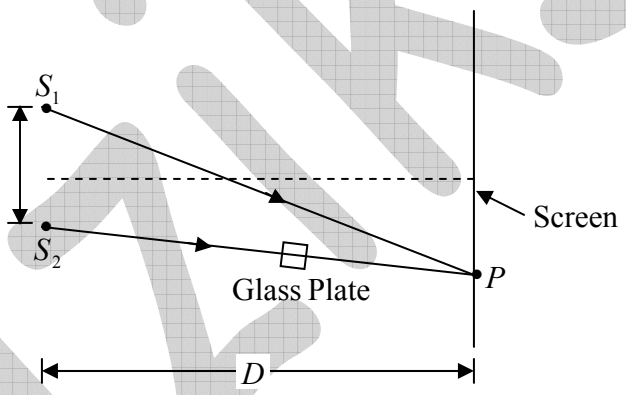
Ans. 13: (d)

Solution: The shift in path when an obstacle of refractive index μ and of length is kept in way of a ray is given as

Path difference $=(\mu-1)t=(1.5-1)30\,mm=15\,mm$

Ans. 14: (b)

Solution:



When a glass plate of width t and refractive index μ is kept in path of one ray say S_2P then the fringe shift n is given as

$$(\mu-1)t=n\lambda \Rightarrow n=\frac{(\mu-1)t}{\lambda}$$

If β is fringe width the lateral shift = $n\beta$

$$\Rightarrow \frac{(\mu-1)t}{\lambda}\beta=\frac{(1.5-1)\times 0.06\times 10^{-3}}{6000\times 10^{-8}}\times 1\times 10^{-3}=5cm$$

MSQ (Multiple Select Questions)

Ans. 15: (b) and (d)

Solution: $\frac{I_{max}}{I_{min}}=\left(\frac{\sqrt{I_1}+\sqrt{I_2}}{\sqrt{I_1}-\sqrt{I_2}}\right)^2$ or $\frac{9}{1}=\left(\frac{\sqrt{I_1}+\sqrt{I_2}}{\sqrt{I_1}-\sqrt{I_2}}\right)^2$

or $3\left(\sqrt{I_1}-\sqrt{I_2}\right)=\sqrt{I_1}+\sqrt{I_2}$ or $2\sqrt{I_1}=4\sqrt{I_2}$ or $\frac{I_1}{I_2}=\frac{4}{1}$

option (b) is correct.

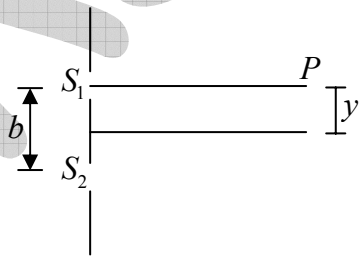
Again $\frac{I_1}{I_2}=\frac{4}{1}$ or $\frac{a^2}{b^2}=\frac{4}{1}$ $\left[\because I\propto(\text{amplitude})^2\right]$ or $\frac{a}{b}=\frac{2}{1}$

Option (d) is correct.

Ans. 16: (a) and (c)

Solution: Those wavelengths are missing for which path difference

$\Delta=(2n-1)\frac{\lambda}{2}$ is directly in front of slit S_1.



$\because \Delta=\frac{yd}{D}$ $\therefore$ $y=(2n-1)\frac{\lambda}{2}\frac{D}{d}$

$\frac{b}{2}=\frac{(2n-1)\lambda}{2b}\left[\because y=\frac{b}{2},d=b,D=d\right]$ $\lambda=\frac{b^2}{(2n-1)d}$ where $n=1,2,3...$

$\therefore$ For $n=1,\lambda_1=\frac{b^2}{d}$, For $n=2,\lambda_2=\frac{b^2}{3d}$, for $n=3,\lambda_3=\frac{b^2}{5d}$

Thus correct answers are (a) and (c).

Ans. 17: (a) and (c)

Solution: Young's experiment is based on division of wavefront. The option (a) is correct.

The central point will be bright point.

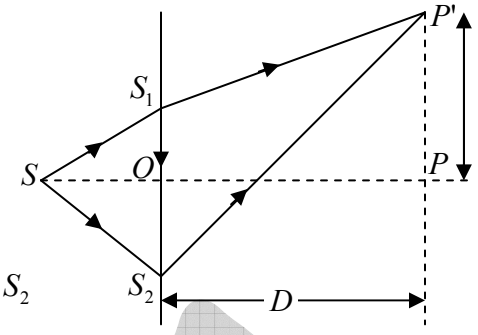
The position on screen where maxima appears is $y_n=n\frac{D\lambda}{d}$ when $n=0,1,2,3$

For $n=0,\ \ y=0$

Which corresponds to central point? Thus central point on the screen has central bright maxima. Therefore option (c) is correct

Ans. 18: (a), (b) and (c)

Solution: Young's double slit experiment:



$S_1, S_2 =$ two slits

$S_1S_2 = d =$ distance between the slits

$S =$ light source

PP' 's screen at a distance D from the slits S_1, S_2

(i) For bright fringe at P'

$$\frac{yd}{D} = n\lambda \qquad n = 0,\ 1,\ 2.....$$

(ii) For dark fringe at P'

$$\frac{yd}{D} = \left(n + \frac{1}{2}\right)\lambda \qquad n = 0,\ 1,\ 2.....$$

Fringe width $\beta = \frac{\lambda D}{d}$

If, we use ordinary extended source of light than initial slit is necessary whereas, if source of light is well collimated, then it is not necessary. Spatially coherent source is also collimated.

Ans. 19: (b), (c), and (d)

Solution: Underwater, the wavelength of light would decrease as $\lambda_{water} = \frac{\lambda_{air}}{\mu_{water}}$. Since the position of light and dark bands are proportional to λ the underwater fringe separation will decrease Thus option (b) is correct

Since water scatter more light therefore the intensity on the screen will be less in water as compare to air. Thus option (c) is correct

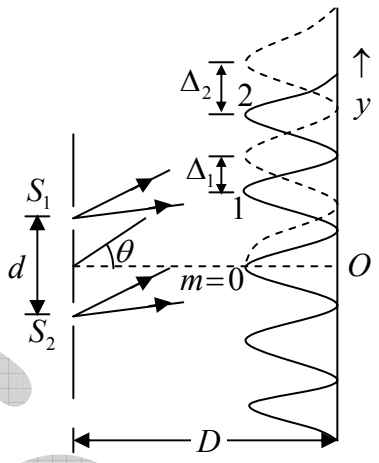
Since visibility is defined as $V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$

If both slits are of same size then $I_{\min} = 0$ and $V = 1$ i.e. 100% visibility. It is irrespective whether slits are in air or in water.

Ans. 20: (a), (b), (c) and (d)

Solution: (a) At position O the path difference is always zero, irrespective of wavelength as path difference $= d\sin\theta$

for central maxima, $\theta = 0$, thus central bright fringes due to λ_1 and λ_2 overlap



(b) For $m = 1, Y_1(\lambda_1) = 1\dfrac{\lambda_1 D}{d} = \dfrac{(420\times10^{-9}\ m)(1)}{(0.025\times10^{-3}\ m)} = 1.68\ cm$

and $Y_1'(\lambda_2) = 1\dfrac{\lambda_2 D}{d} = \dfrac{(520\times10^{-9}\ m)}{0.025\times10^{-3}\ m} = 2.08\ cm$

$\therefore \quad \Delta_1 = Y_1^1 - Y_1 = 2..08 - 1.68 = 0.4\ cm$

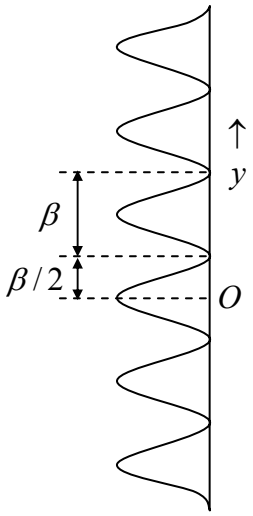
(c) For overlapping, $\dfrac{\lambda_1 D}{d}m = \dfrac{\lambda_2 D}{d}m' \Rightarrow \dfrac{\lambda_1}{\lambda_2} = \dfrac{m'}{m} \therefore \quad \dfrac{m'}{m} = \dfrac{420\ nm}{520\ nm} = \dfrac{42}{52}$

Thus 52nd bright fringe of the $420\ nm$ light overlap with 42nd bright fringe of the $520\ nm$ light

(d) To find the value of y for these fringes $y = 52\times\dfrac{(420\times10^{-9}\ m)(1m)}{0.025\times10^{-3}\ m} = 87.4\ cm$

Ans. 21: (b) and (c)

Solution: The shift in central fringe due to Plexiglass is $\Delta = \dfrac{D(\mu-1)t}{d}$



whereas fringe width is $\beta = \dfrac{D\lambda}{d}$. If the centre point on the screen is to be a dark spot rather than bright, then $\Delta = \dfrac{\beta}{2}, \dfrac{3\beta}{2}, \dfrac{5\beta}{2}, ...$

i.e.

$$\frac{D(\mu-1)t}{d} = \frac{1}{2}\frac{D\lambda}{d} \Rightarrow t = \frac{\lambda}{2(\mu-1)} \quad \frac{D(\mu-1)t}{d} = \frac{3}{2}\frac{D\lambda}{d} \Rightarrow t = \frac{3\lambda}{2(\mu-1)}$$

Thus correct options are (b) and (c).

Ans. 22: (a) and (b)

Solution: The fringe next to the central maxima will be in the order of violet, indigo, blue, green, yellow, orange and red according to their increasing wavelength. Because the position of fringes is directly proportional to wavelength as $\gamma = m\frac{D\lambda}{d}$

(b) The path difference between two interfering waves is

$$\Delta = d\sin\theta$$

At the centre of screen, $\theta = 0$ thus path difference will be zero irrespective of wavelength. Zero path difference leads to interference maxima at the centre of screen for all the wavelength as a result central fringe will be white.

(c) and (d) there will be no dark fringes as some of the maxima's will overlap to minima and give band of non-zero intensity.

Ans. 23: (a), (b) and (c)

Solution: (a) fringe width is $\beta = \frac{\lambda D}{d}$. Since $\lambda_{violet} < \lambda_{red}$, therefore $\beta_{violet} < \beta_{red}$. Thus fringes will be closer.

(b) Violet is monochromatic in nature, thus central fringe will be of violet colour and not white.

(c) Visibility $V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$

Since, there is no change in $I_{\max}$ and $I_{\min}$. Thus visibility remains unchanged.

NAT (Numerical Answer Type Questions)

Ans. 24: 2

Solution: when sources are coherent $I_m = 4I_0$

When sources are incoherent $I = I_0 + I_0 = 2I_0$

$\therefore$ Ratio $= \dfrac{I_m}{I} = \dfrac{4I_0}{2I_0} = 2$.

Ans. 25: 15.8

Solution: Consider two adjacent fringes, namely the mth-order and the next one $(m+1)$, we get

$$y_n = n\frac{\lambda D}{d},\ y_{n+1} = (n+1)\frac{\lambda D}{d}$$

$$\therefore\ \Delta y = y_{n+1} - y_n = \frac{\lambda D}{d} = \frac{(632.8\times10^{-9}\ m)(5m)}{0.2\times10^{-3}\ m} = 1.58\times10^{-2}\ m = 15.8\ mm$$

Ans. 26: 2.61

Solution: Position of dark fringes on screen is

$$y_{dark} = \left(n+\frac{1}{2}\right)\frac{\lambda D}{d} \qquad \text{where } m = 0,1,2,3....$$

The spacing between the first and second dark fringes is

$$\Delta y = \left(1+\frac{1}{2}\right)\frac{\lambda D}{d} - \left(0+\frac{1}{2}\right)\frac{\lambda D}{d} = \frac{\lambda D}{d}$$

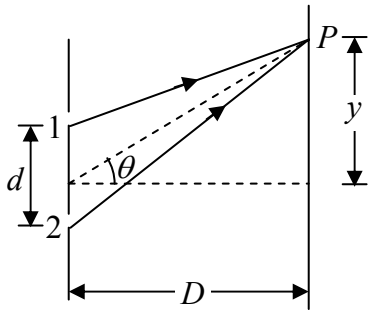
$$D = \frac{(\Delta y)d}{\lambda} = \frac{(4\times10^{-3}\ m)(0.3\times10^{-3}\ m)}{460\times10^{-9}\ m} \Rightarrow D = 2.61\ m$$

Ans. 27: 1.54

Solution: The condition far interference minimum is

$$d\sin\theta = \left(n+\frac{1}{2}\right)\lambda \qquad n = 0,1,2,3....$$

The first minimum is described by $m = 0$ and the tenth by $m = 9$



Now far 10th minimum $\sin\theta = \frac{\lambda}{d}\left(9+\frac{1}{2}\right) \Rightarrow \sin\theta = \frac{9.5\lambda}{d}$ also $\tan\theta = \frac{y}{D}$

but far small angle $\sin\theta \cong \tan\theta$

$$\frac{9.5\lambda}{d} = \frac{Y}{D} \Rightarrow \quad d = \frac{9.5\lambda D}{y}$$

where $\lambda = 589\times10^{-9}\ m, \quad D = 2\ m, \quad y = 7.26\times10^{-3}\ m$

$$\therefore \quad d = \frac{9.5\times\left(589\times10^{-9}\ m\right)\times(2m)}{7.26\times10^{-3}\ m} = 1.54\times10^{-3}\ m \Rightarrow d = 1.54\ mm$$

Ans. 28: 515

Solution: The condition for bright fringes is

$$y_n = n\frac{\lambda D}{d}, \qquad n = 1,2,3,.....$$

For $n=1, \quad y = \frac{\lambda D}{d} \Rightarrow \lambda = \frac{yd}{D}$

where $y = 3.4\times10^{-3}\ m,\ d = 5\times10^{-4}\ m$ and $D = 3.3\ m$

$$\therefore \qquad \lambda = \frac{\left(3.4\times10^{-3}m\right)\left(5\times10^{-4}m\right)}{3.3\ m} = 515\ nm$$

Ans. 29: 6.28

Solution: The relation between phase difference and path difference is $\delta = \frac{2\pi}{\lambda}$ (Path difference)

where path difference $= d\sin\theta = d\times\frac{y}{D}$

$$\therefore \qquad \delta = \frac{2\pi}{\lambda}d\left(\frac{y}{D}\right)$$

where $\lambda = 5\times10^{-7}\ m,\ d = 1.2\times10^{-4}m,\ y = 5\times10^{-3}m$ and $D = 1.2\ m$

$$\therefore \qquad \phi = \frac{2\pi}{\left(5\times10^{-7}m\right)}\left(1.2\times10^{-4}m\right)\left(\frac{5\times10^{-3}m}{1.2m}\right) = 6.28\ \text{rad.}$$

Ans. 30: 48

Solution: The average light intensity at point P is $I = I_{max}\cos^2\left(\frac{\delta}{2}\right)$

where $\delta = \frac{2\pi}{\lambda}\Delta = \frac{2\pi}{\lambda}d\sin\theta$

$\therefore \quad I = I_{max}\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)$

For small θ, $\sin\theta = \frac{y}{D}$

$\therefore I = I_{max}\cos^2\left(\frac{\pi dy}{D\lambda}\right)$

$\Rightarrow \frac{I}{I_{max}} = \cos^2\left(\frac{\pi dy}{D\lambda}\right) \Rightarrow \sqrt{\frac{I}{I_{max}}} = \cos\left(\frac{\pi dy}{D\lambda}\right) \Rightarrow \frac{\pi dy}{D\lambda} = \cos^{-1}\left(\sqrt{\frac{I}{I_{max}}}\right)$

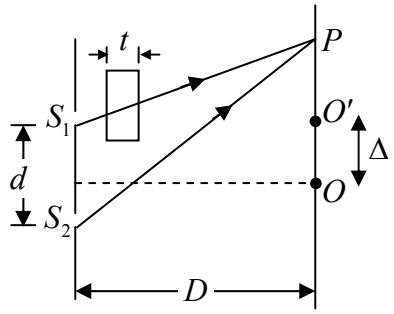
$y = \frac{\lambda D}{\pi d}\cos^{-1}\left(\sqrt{\frac{I}{I_{max}}}\right)$ where $I = 0.75\,I_{max}$, $\lambda = 6\times10^{-7}\,m$, $D = 1.2\,m$ and $d = 2.5\times10^{-3}\,m$

$\therefore y = \frac{\left(6\times10^{-7}\,m\right)\left(1.2\,m\right)}{\pi\left(2.5\times10^{-3}\,m\right)}\cos^{-1}\left(\sqrt{\frac{0.75\,I_{max}}{I_{max}}}\right) = 9.167\times10^{-5}\times\cos^{-1}\left(\sqrt{0.75}\right)$

$= 9.167\times10^{-5}\times\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 9.167\times10^{-5}\times\left(30\times\frac{\pi}{180}\text{rad}\right) = 48.0\times10^{-6}\,m = 48\,\mu m$

Ans. 31: 0.023

Solution: The shift in central maxima due to mica sheet



$\Delta = \frac{D(\mu-1)t}{d} \Rightarrow t = \frac{\Delta d}{D(\mu-1)}$

where $\Delta = 0.4\ cm, d = 0.2\ cm, D = 60\ cm$ and $\mu = 1.58$

$\therefore \quad t = \frac{(0.4\ cm)(0.2\ cm)}{(60\ cm)(1.58-1)} = \frac{0.08}{34.8}\ cm = 2.299\times10^{-3}\ cm$

$= 2.299\times10^{-2}\ mm = 0.023\ mm$

Ans. 32: 9.14

Solution: The shift in central maxima due to mica sheet is $\Delta = \dfrac{D(\mu-1)}{d}$

The fringe width is $\beta = \dfrac{D\lambda}{d}$

According to question $\Delta = 9\beta \Rightarrow \dfrac{D(\mu-1)t}{d} = 9\dfrac{D\lambda}{d} \Rightarrow (\mu-1)t = 9\lambda$ where $\mu = 1.58$ and

$\lambda = 5893\times10^{-10}\ m \ \therefore \quad t = \dfrac{9\times5893\times10^{-10}\ m}{(1.58-1)} = 9.144\times10^{-6}\ m = 9.14\ \mu m$

Ans. 33: 0

Solution: The introduction of mica sheet of thickness t_1 of ray 1 will shift central maxima upward by distance of $\Delta_1 = \dfrac{D(\mu-1)t_1}{d}$ whereas mica sheet of thickness t_2 in the path of ray 2 will shift central maxima down-ward by distance $\Delta_2 = \dfrac{D(\mu-1)t_2}{d}$

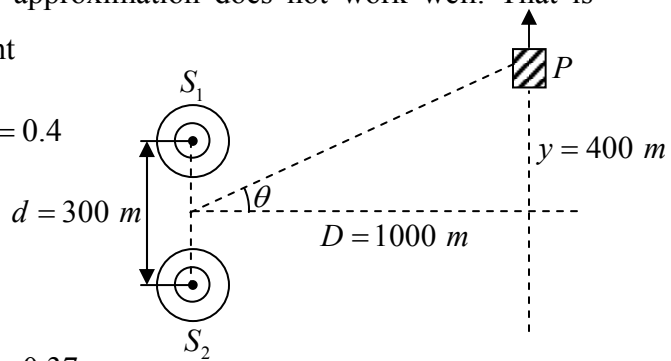
Since thickness $t_1 = t_2 \ \therefore \quad \Delta_1 = \Delta_2$. Thus net shift is $\Delta = \Delta_1 - \Delta_2 = 0$

Ans. 34: 55.7

Solution: The condition of interference maxima is

$$d\sin\theta = n\lambda \Rightarrow \lambda = \frac{d\sin\theta}{n}, \quad n = 0,1,2....$$

with the given condition, the small angle approximation does not work well. That is $\sin\theta, \tan\theta$ and θ are significantly different



At $m = 2$ maximum $\tan\theta = \dfrac{Y}{D} = \dfrac{400\ m}{1000\ m} = 0.4$

$\theta = \tan^{-1}(0.4)$

$\theta = 21.8^{o}$

$\therefore \quad \lambda = \dfrac{d\sin\theta}{m} = \dfrac{(300\ m)(\sin 21.8^{o})}{2} = \dfrac{300\times0.37}{2} \quad \Rightarrow \lambda = 55.7\ m$

Ans. 35: 641

Solution: The condition of interference maxima is $d\sin\theta = m\lambda$

$$\Rightarrow \quad n = \frac{d\sin\theta}{\lambda} \text{ where } d = 0.32\times10^{-3}\ m, \theta = 30^{o} \text{ and } \lambda = 5\times10^{-7}\ m$$

$$\therefore \quad n = \frac{\left(0.32\times10^{-3}\right)\sin\left(30^{o}\right)}{5\times10^{-7}} = \frac{3.2\times10^{-4}\times0.5}{5\times10^{-7}} = 320$$

there are 320 maxima to the right, 320 to the left, and one for $m = 0$ straight ahead. Thus there are 641 maxima

Ans. 36: 2.62

Solution: The condition far bright fringe is $y_n = \frac{m\lambda D}{d}, \quad n = 0,1,2,3...$

For $n = 1$, $y = \frac{\lambda D}{d}$ where $\lambda = 546.1\times10^{-9}\ m, \quad D = 1.2\ m, \quad d = 0.25\times10^{-3}\ m$

$$\therefore \quad y = \frac{\left(546.1\times10^{-9}\ m\right)\left(1.2\ m\right)}{\left(0.25\times10^{-3}\ m\right)} = 2.62\times10^{-3}\ m \Rightarrow y = 2.62\ mm$$

Ans. 37: 14

Solution: The condition far maxima is $Y_n = n\frac{\lambda D}{d}, \quad n = 0,1,2,....$ with $n = 3$, we find that the fringe positions corresponding to these two wavelengths are

$$y = 3\frac{\lambda D}{d} = 3\frac{\left(430\times10^{-9}\ m\right)\left(1.5\ m\right)}{0.025\times10^{-3}\ m} = 7.74\times10^{-2}\ m$$

$$y' = \frac{3\lambda' D}{d} = 3\frac{\left(510\times10^{-9}\ m\right)\left(1.5\ m\right)}{0.025\times10^{-3}\ m} = 9.18\times10^{-3}\ m$$

Hence the separation distance between the two fringes is

$$\Delta y = y' - y = 9.18\times10^{-2}\ m - 7.74\times10^{-2}\ m = 1.4\times10^{-2}\ m$$

$$\Delta y = 1.4\ cm \quad \text{or} \quad \Delta y = 14\ mm$$

Ans. 38: 1.93

Solution: The path difference is $\Delta = d\sin\theta$

for $D >> y$, $\sin\theta \cong \tan\theta = \dfrac{y}{D}$

$$\therefore \quad \Delta = \frac{yd}{D}$$

where, $y = 1.8\times10^{-2}\ m$, $d = 1.5\times10^{-4}\ m$ and $D = 1.4\ m$

$$\therefore \quad \Delta = \frac{\left(1.8\times10^{-2}\ m\right)\left(1.5\times10^{-4}\ m\right)}{1.4\ m} = 1.93\times10^{-6}\ m \quad \Rightarrow \Delta = 1.93\ \mu m$$

fiziks

3. Diffraction of Light

The spreading out of a wave when it passes through a narrow opening is usually referred to as diffraction, and the intensity distribution on the screen is known as the diffraction pattern.

The diffraction phenomena are usually divided into two categories: ***Fresnel diffraction*** and ***Fraunhofer diffraction***.

In the Fresnel class of diffraction the source of light and the screen are, in general, at a finite distance from the diffracting aperture In the Fraunhofer class of diffraction, the source and the screen are at infinite distances from the aperture; this is easily achieved by placing the source on the focal plane of a convex lens and placing the screen on the focal plane of another convex lens. The two lenses effectively moved the source and the screen to infinity because the first lens makes the light beam parallel and the second lens effectively makes the screen receive a parallel beam of light.

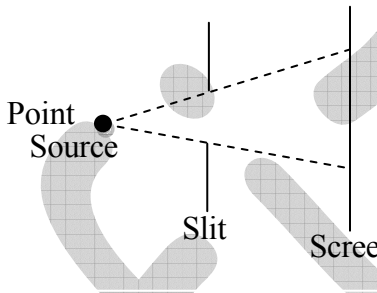


Fig: Fresnel diffraction

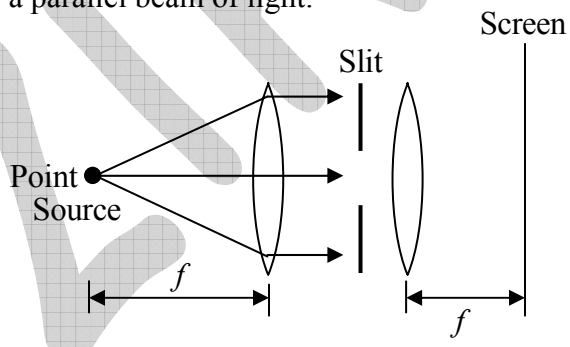
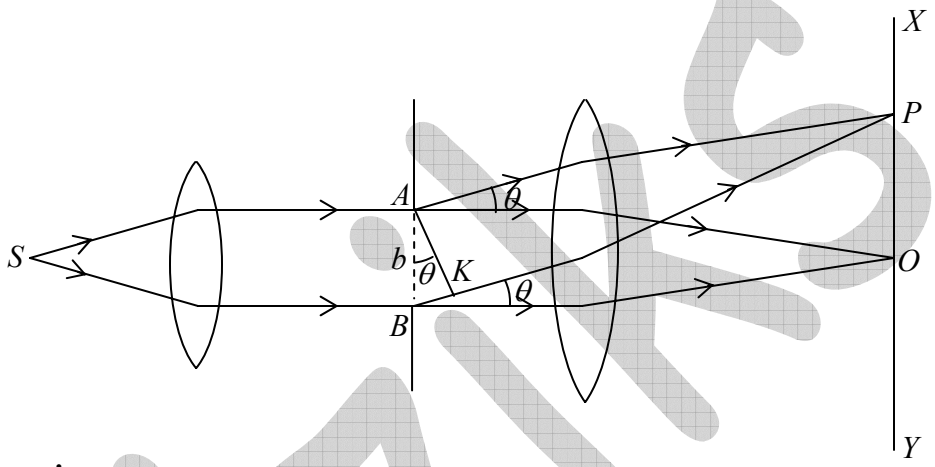


Fig: Fraunhofer diffraction

3.1 Fraunhofer's diffraction at a single slit

Let a parallel beam of monochromatic light of wavelength λ be incident normally upon a narrow slit of width $AB = b$ placed perpendicular to the plane of the paper. Let the diffracted light be focused by a convex lens L on a screen XY placed in the focal plane of the lens. The diffraction pattern obtained on the screen consists of a central bright band, having alternative dark and weak bright bands of decreasing intensity on both sides.



Explanation

In terms of wave theory, a plane wave front is incident normally on the slit AB. According to the Huygen's principle, each point on AB sends out secondary wavelets in all directions. The rays proceeding in the same direction as the incident rays are focused at O, while those diffracted through an angle θ are focused at P. Let us find out the resultant intensity at P.

Let AK be the perpendicular to BK. As the optical path from the plane AK to P are equal, the path difference between the wavelets from A to B in the direction

$$\text{Path difference} = BK = AB\sin\theta = b\sin\theta$$

The corresponding phase difference $= \dfrac{2\pi}{\lambda}(b\sin\theta)$

Let the width AB of the slit is divided into n equal parts. The amplitude of vibration at P. Due to the waves from each part will be the same, say equal to a. The phase difference between the waves from any two consecutive parts is $\frac{1}{n}\left(\frac{2\pi}{\lambda}b\sin\theta\right)=\delta$ (say).

Hence the resultant amplitude at P is $R=\dfrac{a\sin\left(\frac{n\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)}=\dfrac{a\sin\left(\frac{\pi b\sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi b\sin\theta}{n\lambda}\right)}$

Let $\dfrac{\pi b\sin\theta}{\lambda}=\alpha \quad \Rightarrow R=\dfrac{a\sin\alpha}{\sin\left(\frac{\alpha}{n}\right)}=\dfrac{a\sin\alpha}{\frac{\alpha}{n}} \qquad \because \dfrac{\alpha}{n}$ is small

$$R=\frac{n.a.\sin\alpha}{\alpha}$$

As $n\to\infty, \quad a\to 0$ but the product na remains finite

Thus Resultant Amplitude at P due is $R=A\dfrac{\sin\alpha}{\alpha}$ let $na=A$

Thus resultant intensity at P ; $I=R^2=A^2\left(\dfrac{\sin\alpha}{\alpha}\right)^2$

The constant of proportionality being taken as unity for simplicity

Condition for Maximum and Minimum intensity

Direction for Minimum Intensity

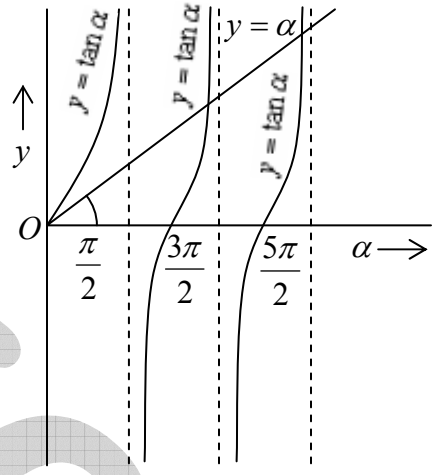
For minimum intensity $I=0 \Rightarrow \dfrac{\sin\alpha}{\alpha}=0 \Rightarrow \sin\alpha=0$ but $\alpha\neq 0$.

Thus $\alpha=\pm m\pi$ where, m has an integral value $1,2,3$ except zero

$$\frac{\pi b\sin\theta}{\lambda}=\pm m\pi$$

$$\Rightarrow b\sin\theta=\pm m\lambda \qquad \text{where} \quad m=1,2,3$$

Direction for Maximum Intensity



For direction of maximum intensity $\frac{dI}{d\alpha}=0$

$$\Rightarrow \frac{d}{d\alpha}\left[A^2\frac{\sin^2\alpha}{\alpha^2}\right]=0 \Rightarrow A^2\left(\frac{2\sin\alpha}{\alpha}\right)\frac{a\cos\alpha-\sin\alpha}{\alpha^2}=0$$

$$\Rightarrow \frac{\alpha\cos\alpha-\sin\alpha}{\alpha^2}=0 \Rightarrow 2\cos\alpha-\sin\alpha=0 \Rightarrow \quad \alpha=\tan\alpha$$

The equation is solved graphically by plotting the curve

$$y=\alpha \quad \text{and} \quad y=\tan\alpha$$

the points of intersection gives (approximately)

$$\alpha=0,\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}=0,1.430\pi,2.462\pi,3.471\pi.....$$

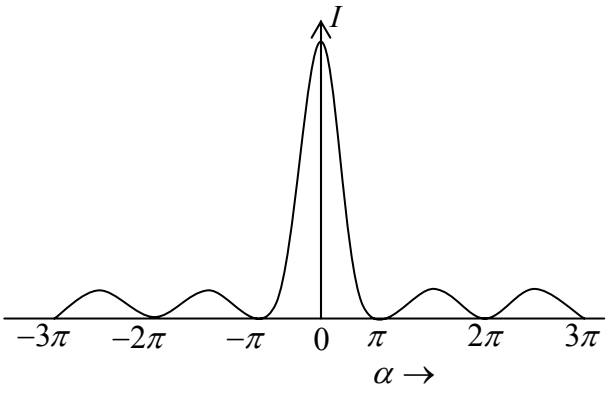
Substituting the value of α into the expression of I

The intensity of the central maxima $I_0=A^2\left(\frac{\sin 0}{0}\right)=A^2$

The intensity of the first maxima $I_1=A^2\left[\frac{\sin\frac{3\pi}{2}}{\frac{3\pi}{2}}\right]^2=\frac{A^2}{22}$

The intensity of the second maxima

$$I_2=A^2\left[\frac{\sin\frac{5\pi}{2}}{\frac{5\pi}{2}}\right]^2=\frac{A^2}{61}$$



Thus the ratio of intensities of the successive maxima

$$I_0,I_1,I_2....=1:\frac{1}{22}:\frac{1}{61}:\frac{1}{121}$$

Clearly most of the incident light is concentrated in the central maxima which occurs in the direction $\alpha=0$ or $\frac{\pi b\sin\theta}{\lambda}=0$ or $\theta=0$ i.e. in the same direction as the incident light

Effect of Slit Width on Diffraction Pattern

If slit is made narrower

Since first minimum on either side of the central maximum occurs in the direction θ, given by equation $b\sin\theta = \pm\lambda$

When the slit is narrowed (b is reduced), the angle θ increases which means the central maximum becomes wider. When the slit-width is as small as wavelength $(b = \lambda)$, the first minimum occurs at $\theta = 90^0$, which means central maxima fills the whole space i.e. condition of uniform illumination.

Example: A monochromatic light with a wavelength of $\lambda = 600\,nm$ passes through a single slit which has a width of $0.800\,mm$.

(a) What is the distance between the slit and the screen be located if the first minimum in the diffraction pattern is at a distance $1.00mm$ from the center of the screen?

(b) Calculate the width of the central maximum.

Solution: (a) The general condition for destructive interference is

$$\sin\theta = m\frac{\lambda}{b} \qquad m = \pm1, \pm2, \pm3, \ldots$$

For small θ, we employ the approximation $\sin\theta \approx \tan\theta = \frac{y}{D}$, which yields $\frac{y}{D} \approx m\frac{\lambda}{b}$

The first minimum corresponds to $m = 1$. If $y_1 = 1.00\,mm$, then

$$L = \frac{by_1}{m\lambda} = \frac{\left(8.00\times10^{-4}\,m\right)\left(1.00\times10^{-3}\,m\right)}{1\left(600\times10^{-9}\,m\right)} = 1.33\,m$$

(b) The width of the central maximum is $W = 2y_1 = 2\left(1.00\times10^{-3}\,m\right) = 2.00\,mm$

Example: A monochromatic light is incident on a single slit of width $0.800\,mm$ and a diffraction pattern is formed at a screen which is $0.800\ m$ away from the slit. The second order bright fringe is at a distance $1.60\,mm$ from the centre of the central maximum. What is the wavelength of the incident light?

Solution: The general condition for destructive interference is

$$\sin\theta = m\frac{\lambda}{b} \approx \frac{y}{D}$$ where small-angle approximation has been made.

Thus the position of the mth order dark fringe measured from the central axis is

$$y_m = m\frac{\lambda D}{b}$$

Let the second bright fringe be located halfway between the second and the third dark fringe. That is $y_{2b} = \frac{1}{2}(y_2 + y_3) = \frac{1}{2}(2+3)\frac{\lambda D}{b} = \frac{5\lambda D}{2b}$

The approximate wavelength of the incident light is then

$$\lambda = \frac{2by_{2b}}{5D} = \frac{2\left(0.800\times10^{-3}\,m\right)\left(1.60\times10^{-3}\,m\right)}{500\left(0.800\,m\right)} = 6.40\times10^{-7}\,m$$

Example: Light of wavelength $580\ nm$ is incident on a slit having a width of $0.30\ mm$. The viewing screen is $2\ m$ from the slit. Find the positions of the first dark fringes and the width of the central bright fringe. What if the slit width is increased by an order of magnitude of $3.0\ mm$? What happens to the diffraction pattern?

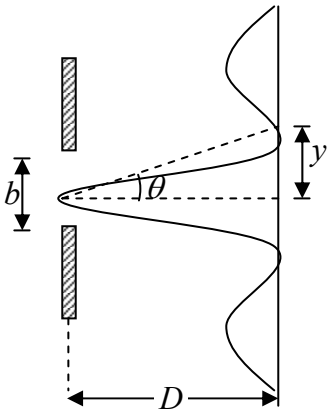
Solution: To analyze the problem, note that the two dark fringes that flank the central bright fringe corresponds to

$$n = \pm 1\,.$$

$$\therefore \qquad b\sin\theta_{dark} = \pm\frac{\lambda}{b}$$

$$\Rightarrow \qquad \sin\theta_{dark} = \pm\frac{5.80\times10^{-7}\,m}{0.3\times10^{-3}\,m} = \pm1.933\times10^{-3}$$

since, $y_1 = D\tan\theta_{dark}$



For small angle $\theta_{dark} \cong \tan\theta_{dark} \cong \sin\theta_{dark} \cong \dfrac{y_1}{D}$

$\therefore y_1 = D\sin\theta_{dark} = (2m)\left(\pm 1.933\times10^{-3}\right) = \pm 3.87\times10^{-3}\, m$

The width of the central bright fringe is $W = 2|y_1| = 2\times3.87\times10^{-3}\, m = 7.74\ mm$

Note that this value is much greater than the width of the slit.

We expect that angles at which the dark bands appear will decrease as b increases

Thus the diffraction pattern narrows.

For $b = 3.0\ mm,$ the sine of the angle θ_{dark} for the $n = \pm1$ dark fringes are

$$\sin\theta_{dark} = \pm\frac{\lambda}{b} = \pm\frac{5.8\times10^{-7}\ m}{3\times10^{-3}\ m} = \pm1.933\times10^{-4}$$

$$\therefore \quad y_1 \cong D\sin\theta_{dark} = (2.0\ m)\left(\pm1.933\times10^{-4}\right) = \pm3.87\times10^{-4}\ m$$

The width of the central bright fringe is

$$W = 2y_1 = 2\times3.87\times10^{-4}\ m$$

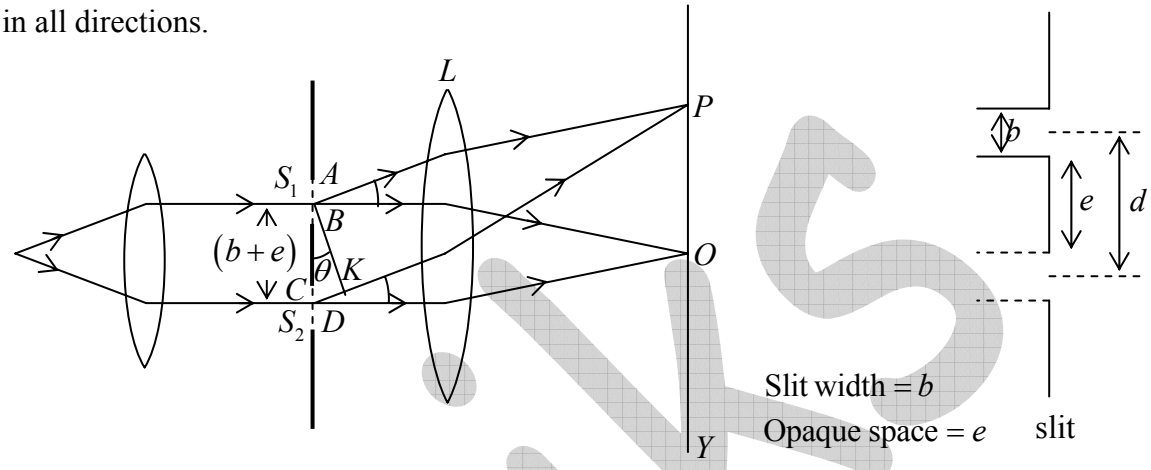
$$\Rightarrow \quad W = 0.774\ mm$$

Notice that this is smaller than the width of the slit

3.2 Fraunhofer's Diffraction at a Double Slit

In Double Slit diffraction experiment, the pattern obtained on the screen is the diffraction pattern due to a single slit on which a system of interference fringes is superposed.

By Huygen's principle every point in the slits AB and CD sends out secondary wavelets in all directions.



From the theory of diffraction at a single slit, the resultant amplitude due to wavelets diffracted from each slit in a direction θ is $R = A\dfrac{\sin\alpha}{\alpha}$, where $\alpha = \dfrac{\pi b \sin\theta}{\lambda}$.

Consider the two slits as equivalent to two coherent sources placed at the middle points S_1 and S_2 of the slits and each sending wavelets of amplitude $A\dfrac{\sin\alpha}{\alpha}$ in a direction θ.

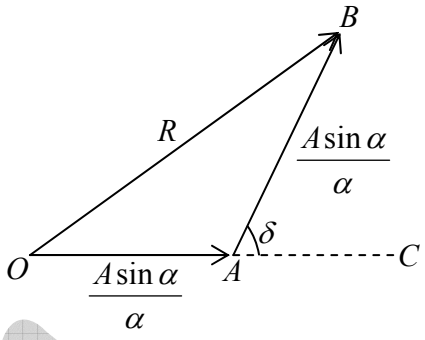
Consequently, the resultant amplitude at a point P on the screen will be the result of interference between two waves of same amplitude $A\dfrac{\sin\alpha}{\alpha}$, and having a phase difference δ (say) and path difference $S_2K = (b+e)\sin\theta$.

Phase difference $\delta = \dfrac{2\pi}{\lambda}(b+e)\sin\theta$

The resultant amplitude R

$$R^2 = \left(\frac{A\sin\alpha}{\alpha}\right)^2 + \left(\frac{A\sin\alpha}{\alpha}\right)^2 + 2\left(\frac{A\sin\alpha}{\alpha}\right)\left(\frac{A\sin\alpha}{\alpha}\right)\cos\delta$$

$$R^2 = \left(A\frac{\sin\alpha}{\alpha} \right)^2 (2 + 2\cos\delta) \Rightarrow R^2 = \frac{A^2\sin^2\alpha}{\alpha^2} 4\cos^2\frac{\delta}{2}$$



Let $\beta = \frac{\delta}{2} \Rightarrow \beta = \frac{\pi(b+e)\sin\theta}{\lambda}$

Therefore the resultant intensity at P is

$$I = R^2 = 4A^2\frac{\sin^2\alpha}{\alpha^2}\cos^2\beta$$

Thus the intensity in the resultant pattern depends on two factors

(i) $\frac{\sin^2\alpha}{\alpha^2}$, which gives diffraction pattern due to each individual slit and

(ii) $\cos^2\beta$, which gives interference pattern due to diffracted light waves from the two slits.

The diffraction term $\frac{\sin^2\alpha}{\alpha^2}$ gives central maximum in the direction $\theta = 0$, having alternate minima and subsidiary maxima of decreasing intensity on either side the minima are obtained in the direction given by

$$\sin\alpha = 0 \quad \Rightarrow \quad \alpha = \pm m\pi$$

$$\frac{\pi b\sin\theta}{\lambda} = \pm m\pi \qquad \text{where } m = 1, 2, 3, ... \text{ (but not zero)}$$

Thus the condition of diffraction minimum is

$$b\sin\theta = \pm m\lambda \qquad (m = 1, 2, 3, ...)$$

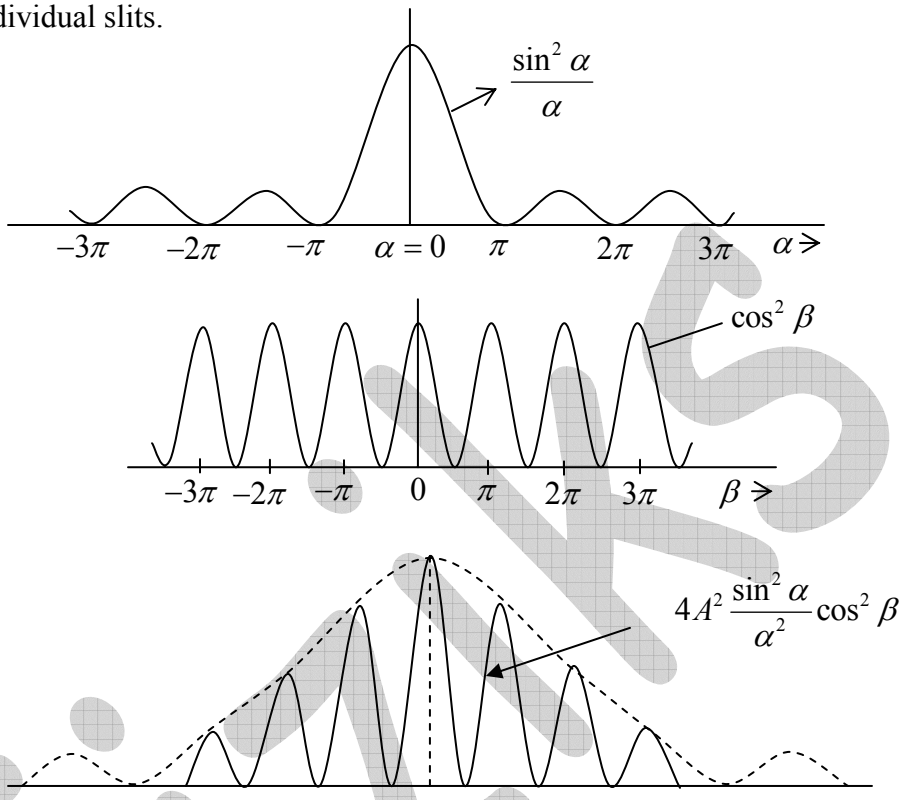
The interference term $\cos^2\beta$ gives a set of equidistant dark and a bright fringe as in Young's double slit interference experiment. The bright fringe are obtained in the direction

$$\cos^2\beta = 1 \quad \Rightarrow \beta = \pm n\pi \quad \Rightarrow \quad \frac{\pi}{\lambda}(b+e)\sin\theta = \pm n\pi$$

Thus the condition of maximum is

$$(b+e)\sin\theta = \pm n\lambda \qquad \text{where } n = 0, 1, 2, ...$$

The entire pattern may be considered as consisting of interference fringes due to light from both slits, the intensities of these fringes being governed by the diffraction occurring at the individual slits.



Effect of Slit Width on Diffraction Pattern

(1) Effect of increasing the slit width

If we increase the slit width b, the envelope of the fringe pattern changes so that it central peak is sharper. The fringe spacing, which depends on slit separation, does not change. Hence less interference maxima now fall within the central diffraction maximum.

(2) Effect of increasing the distance between slits

If b is kept constant and the separation d between them is increased, the fringes becomes closer together, the envelope of the pattern remaining unchanged. Thus more interference maxima fall within the central envelope.

Absent Orders

For certain values of d certain interference maxima become absent from the pattern. Suppose for some value of θ the following conditions are simultaneously satisfied.

$$(b+e)\sin\theta = \pm n\lambda \qquad \text{Interference maxima}$$

$$b\sin\theta = \pm m\lambda \qquad \text{Diffraction minima}$$

Thus according to the first condition there should be an interference maximum in the direction θ, but according to the second condition there is no diffracted light in this direction. Therefore, the interference maximum will be absent in this direction.

$$\frac{b+e}{b} = \frac{n}{m}$$

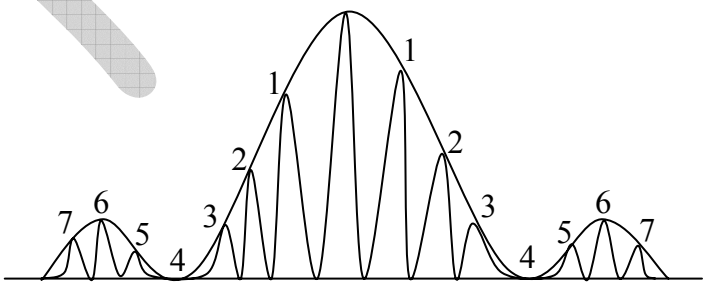
(i) If $e = b$, then $n = 2m = 2, 4, 6, ...$ Since $m = 1, 2, 3, ...$

Thus 2nd, 4th and 6th order interference maxima will be absent, i.e. they will coincide with 1st, 2nd, 3rd order diffraction minima. Thus the central diffraction maximum will have three interference (zero order and two first orders) maxima.

(ii) If $e = 2b$, then $n = 3m = 3, 6, 9, ...$

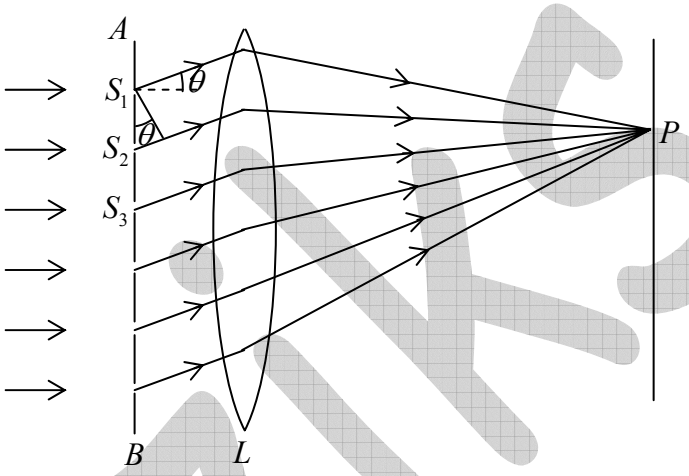
Thus the 3rd, 6th, 9th order interference maxima will coincide with 1st, 2nd, 3rd order diffraction minima. Thus the central diffraction maximum will have five interference maxima.

(iii) If $e = 3b$, then $n = 4m = 4, 8, 12, ...$ that is 4th, 8th, 12th ……. order interference maxima will coincide with 1st, 2nd and 3rd order diffraction minima. Thus the central diffraction maximum will have seven inference maxima as shown below.



3.3 Diffraction Grating

A diffraction grating is an arrangement equivalent to a large number of parallel slits of equal widths and separated from one-another by equal opaque spaces. It is made by ruling a large number of fine, equidistant and parallel lines on an optically plane glass plate with a diamond point. The rulings scatter the light and are effectively opaque while the un-ruled parts transmit light and act as slits.



Let AB be the section of a plane transmission grating, the lengths of the slits being perpendicular to the plane of the paper. Let b be the width of each slit and e the width of each opaque space between the slits, then $d = b + e$ is called the grating element.

Let a parallel beam of monochromatic light of wavelength λ be incident normally on the grating. By the theory of Fraunhofer diffraction at a single slit, the wavelets from all points in a slit diffracted in a direction θ are equivalent to a single wave of amplitude $A\frac{\sin\alpha}{\alpha}$, starting from the middle point of the slit where $\alpha = \frac{\pi b \sin\theta}{\lambda}$.

Thus if N be total number of slits in the grating, the diffracted rays from all the slits are equivalent to N parallel rays, one each from the middle points $S_1, S_2, S_3 \ldots$ of the slits. Path difference between the rays from the slits S_1 and S_2 is

$$S_2K = S_1S_2 \sin\theta = (b+e)\sin\theta$$

The corresponding phase difference $= \frac{2\pi}{\lambda}(b+e)\sin\theta = 2\beta$

Hence the resultant amplitude in the direction θ is $R = A\frac{\sin\alpha}{\alpha}.\frac{\sin N\beta}{\sin\beta}$

The resultant intensity is $I = R^2 = A^2\left(\frac{\sin^2\alpha}{\alpha^2}\right)\left(\frac{\sin^2 N\beta}{\sin^2\beta}\right)$

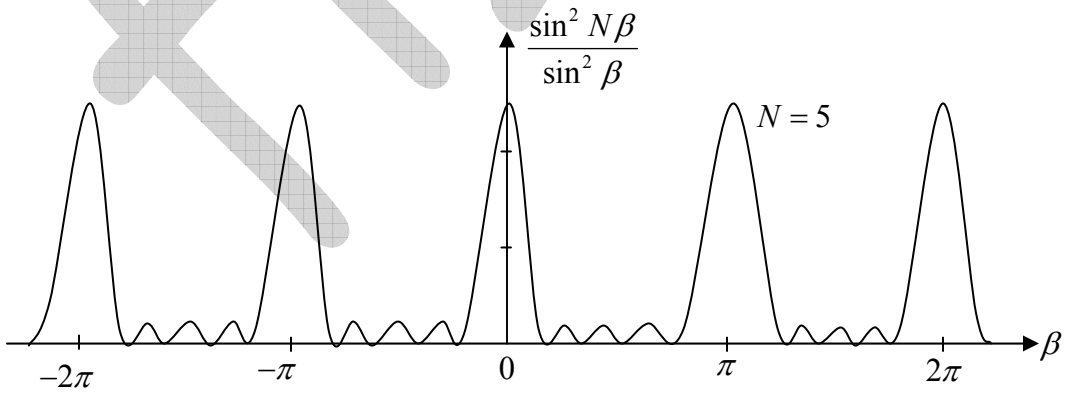
The first term represents the diffraction pattern produced by a single slit whereas the second term represents the Interference pattern produced by N equally spaced slits. For $N=1$ the above equation reduces to a single slit diffraction pattern and for $N=2$, to the double-slit diffraction pattern.

Following figure shows the plot of $\frac{\sin^2 N\beta}{\sin^2\beta}$ as a function of β for $N=5$. As the value of N becomes very large, the above function becomes narrow and very sharply peaked at $\beta = 0, \pi, 2\pi....$

Between the two peaks, the function vanishes when

$$\beta = \frac{m'\pi}{N} \qquad m' = +1, +2, +3, ... \text{ but } m' \neq 0, +N, +2N.....$$

which are referred to as interference minima.



Position of Maxima and Minima

Condition for Principle Maxima

When the value of N is very large, one obtains intense maxima when $\sin\beta = 0$. This condition arises when

$$\beta = \pm n\pi \qquad \text{where } n = 0,1,2,3.....$$

so $\sin N\beta = 0$

Thus $\dfrac{\sin N\beta}{\sin\beta} = \dfrac{0}{0}$ (indeterminate form), thus

$$\lim_{\beta\to\pm n\pi}\frac{\sin N\beta}{\sin\beta} = \lim_{\beta\to\pm n\pi}\frac{N\cos N\beta}{\cos\beta} = \pm N$$

Thus intensity

$$I = \frac{A^2\sin^2\alpha}{\alpha^2}N^2 = N^2 I_0\frac{\sin^2\alpha}{\alpha^2}$$

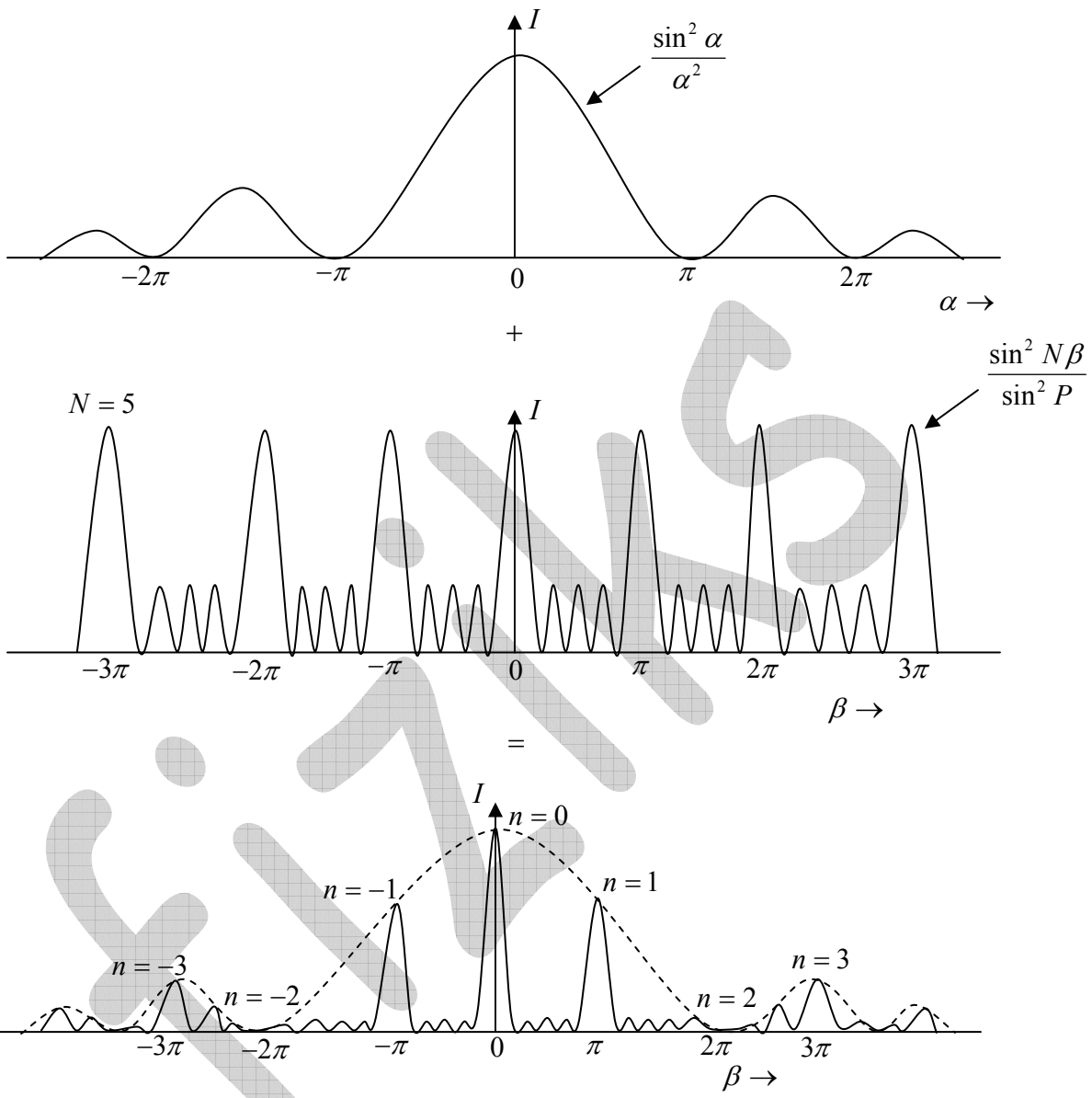
Such maxima are most intense and are known as principal maxima. Physically, at these maxima the fields produced by each of the slits are in phase, and therefore, they add and the resultant field is N times the field produced by each of the slits.

They are obtained in the direction given by $\beta = \pm n\pi$

$$\frac{\pi}{\lambda}(b+e)\sin\theta = \pm n\pi$$

$$(b+e)\sin\theta = \pm n\lambda \qquad \text{where } n = 0,1,2...$$

This is the condition of maxima where for $n = 0$, we get the zero order principle maximum. For $n = \pm 1, +2 + 3.....$ we obtain the first, second, third- order principal maxima respectively. The $\pm$ sign show that there are two principal maxima for each order lying on either side of the zero order maximum.



Condition for Minima

The intensity is zero when $\sin N\beta = 0$ but $\sin\beta \neq 0$ then $\dfrac{\sin N\beta}{\sin\beta} = 0$

($\sin\beta = 0$ gives maximum intensity). Thus minimum are obtained in the directions given by

$$\sin N\beta = 0 \quad \Rightarrow \quad N\beta = \pm m'\pi \qquad \Rightarrow N\frac{\pi}{\lambda}(b+e)\sin\theta = \pm m'\pi$$

Thus the condition of minimum intensity is

$$N(b+e)\sin\theta = \pm m'\lambda$$

where m' takes all integral values except $0, N, 2N.... nN$. Because these value of m' makes $\sin\beta = 0$, which gives principal maxima.

It is clear from above that $m' = 0$ gives the principal maximum and $m' = 1,2,3..$ $(N-1)$ give minima and then $m' = N$ give again a principal maximum. Thus there are $(N-1)$ minima between two consecutive principal maxima.

Condition for Secondary Maxima

As there are $(N-1)$ minima between two consecutive principal maxima, there must be $(N-2)$ other maxima between two principal maxima these are called secondary maxima. Their positions are obtained by differentiating the formula of intensity I with respect to β and equating it equal to zero. Thus

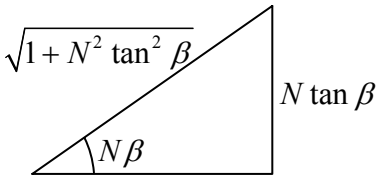
$$\frac{dI}{d\beta} = \frac{A^2\sin^2\alpha}{\alpha^2}2\left[\frac{\sin N\beta}{\sin\beta}\right]\frac{N\cos N\beta\sin\beta - \sin N\beta\cos\beta}{\sin^2\beta} = 0$$

$$N\cos n\beta\sin\beta = \sin N\beta\cos\beta \qquad \Rightarrow \tan N\beta = N\tan\beta \qquad \text{----(1)}$$

To find the value of $\frac{\sin^2 N\beta}{\sin^2\beta}$ under the condition (1), we make use of the triangle shown in figure (1), this gives

$$\frac{\sin^2 N\beta}{\sin^2\beta} = \frac{N^2\tan^2\beta}{(1+N^2\tan^2\beta)} = \frac{N^2}{(1+N^2\tan^2\beta)\cos^2\beta}$$

$$\frac{N^2}{\cos^2\beta + N^2\sin^2\beta} = \frac{N^2}{1+(N^2-1)\sin^2\beta}$$



1

This gives $\sin N\beta = \frac{N\tan\beta}{\sqrt{1+N^2\tan^2\beta}}$

This shows that the intensity of the secondary maxima is proportional to $\frac{N^2}{1+\left(N^2-1\right)\sin^2\beta}$, where the intensity of the principal maxima is proportional to N^2 so,

$$\frac{\text{Intensity of secondary maxima}}{\text{Intensity of principal maxima}}=\frac{1}{1+\left(N^2-1\right)\sin^2\beta}$$

Hence greater the value of N, the weaker are secondary maxima. In an actual grating, N is very large. Hence these secondary maxima are not visible in the grating spectrum.

Condition for absent spectra

A particular principal maximum may be absent if it corresponds to the angle which also determines the minimum of the single-slit diffraction pattern.

Principal maxima in the grating spectrum are obtained in the direction given by

$$\left(b+e\right)\sin\theta=n\lambda \qquad \text{-----(1)}$$

where n is the order of maximum.

The minima in a single slits pattern are obtained in the direction given by

$$b\sin\theta=m\lambda \qquad m=1,2,3... \qquad \text{-----(2)}$$

If both (1) and (2) are satisfied simultaneously, a particular maximum of order n will be missing in the grating spectrum.

$$\frac{b+e}{b}=\frac{n}{m}$$

this is the condition for the spectrum of the order n to be absent.

If $e=b$ then $n=2m=2,4,6...$ i.e. 2nd, 4th, 6th, order spectra will be absent

If $e=2b$ then $n=3m=3,6,9...$ i.e. 3rd, 6th, 9th…… order spectra will be absent

Example: Monochromatic light beam a helium-neon laser $(\lambda = 632.8\ nm)$ is incident normally on a diffraction grating containing 6000 grooves per centimeter. Find the angles at which the 1st and 2nd order maxima are observed. What if we look for the 3rd order maxima? Do we find it?

Solution: First we must calculate the slit separation, which is equal to the inverse of the number of grooves per centimeter.

$$d = \frac{1}{6000}\ cm = 1.667 \times 10^{-4}\ cm = 1667\ nm$$

For the 1st order maximum $(n = 1)$, we obtain

$$\sin\theta_1 = \frac{\lambda}{d} = \frac{632.8\ nm}{1667\ nm} = 0.3796 \Rightarrow \theta_1 = 22.31^{o}$$

For the 2nd order maximum $(n = 2)$, we find

$$\sin\theta_2 = \frac{2\lambda}{d} = \frac{2(632.8\ nm)}{1667\,nm} = 0.7592 \Rightarrow \theta_2 = 49.39^{o}$$

For 3rd order maximum $(n = 3)$, we find

$$\sin\theta_3 = \frac{3\lambda}{d} = \frac{3(632.8\ nm)}{1667\ nm} = 1.139$$

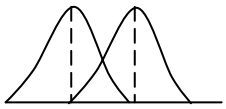
because $\sin\theta$ cannot exceed unity, this does not represent a realistic solution. Hence only zeroth 1st & 2nd order maxima are observed for this situation

3.4 Rayleigh Criterion of Resolution and Resolving Power

The resolving power of an optical instrument represents its ability to produce distinctly separate spectral lines of light having two or more close wavelengths.

3.4.1 Rayleigh's Criterion of Resolution

Lord Rayleigh proposed the following criterion for resolution which has been universally adopted. "Two spectral lines of equal intensities are just resolved by an optical instrument when the principal maximum of the diffraction pattern due to one falls on the first minimum of the diffraction pattern of the other.



3.4.2 Resolving Power of a Grating

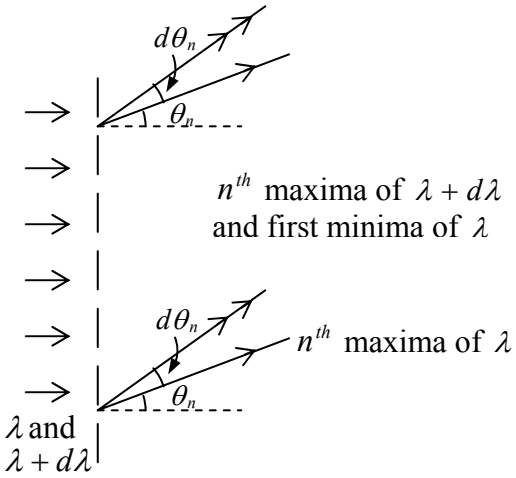
The resolving power of a grating represents its ability to form separate spectral lines for wavelengths very close together. It is defined by $R = \frac{\lambda}{\Delta\lambda}$ where $\Delta\lambda$ is the separation of the two wavelengths which the grating can just resolve; the smaller the value of $\Delta\lambda$, larger the resolving power.

Let a parallel beam of light of two wavelengths of λ and $\lambda + d\lambda$ be incident normally on the grating. If the nth principal maximum of λ is formed in the direction θ_n, we have

$$(b+e)\sin\theta_n = n\lambda \qquad (1)$$

where $(b+e)$ is the grating element.

Let the first minimum adjacent to the nth maximum be obtained in the direction $(\theta_n + d\theta_n)$.



The equation for the minima is

$$N(b+e)\sin\theta = m'\lambda \qquad (2)$$

here $\theta = \theta_n + d\theta_n$

where N is the total number of rulings on the grating and m' takes all integral values except $0, N, 2N, ...nN$ because these values of m' give $0^{th}, 1^{st}, 2^{nd}, ...n^{th}$ principal maximum respectively.

Clearly, the first minimum adjacent to the principal maximum in the direction of θ increasing will be obtained for $m' = (nN+1)$. Therefore, if this minimum is obtained in the direction $\theta_n + d\theta_n$, we have from equation (2)

$$N(b+e)\sin(\theta_n + d\theta_n) = (n.N+1)$$

$$(b+e)\sin(\theta_n + d\theta_n) = \frac{n.N+1}{N}\lambda \qquad (3)$$

By Rayleigh's criterion, the wavelengths λ and $(\lambda + d\lambda)$ are just resolved by the grating when the nth maximum of $(\lambda + d\lambda)$ is also obtained in the direction $\theta_n + d\theta_n$. Then, we have from equation (1)

$$(b+e)\sin(\theta_n + d\theta_n) = n(\lambda + d\lambda) \qquad (4)$$

Comparing equation (3) and (4), we get

$$\frac{n.N+1}{N}\lambda = n(\lambda + d\lambda)$$

$$n.N\lambda + \lambda = Nn\lambda + + Nnd\lambda$$

$$\lambda = Nnd\lambda$$

$$\frac{\lambda}{d\lambda} = nN$$

But $\frac{\lambda}{d\lambda}$ is the resolving power R of the grating.

Therefore, $R = \frac{\lambda}{d\lambda} = nN$

The above expression may be written as

$$R = nN = \frac{N(b+e)\sin\theta_n}{\lambda}$$

But $N(b+e)$ is the total width of the grating. Hence at a particular angle of diffraction θ_n, the Resolving Power is directly proportional to the total width of the ruled space on the grating.

3.4.3 Difference between Dispersive Power and Resolving Power

The dispersive power of a diffraction grating gives us an idea of the angular separation between the lines of a spectrum produced by a grating, it is measured by $\frac{d\theta}{d\lambda}$ where $d\theta$ is the angular separation between two spectral lines whose wavelengths differ by $d\lambda$. The value of Dispersive Power is given by

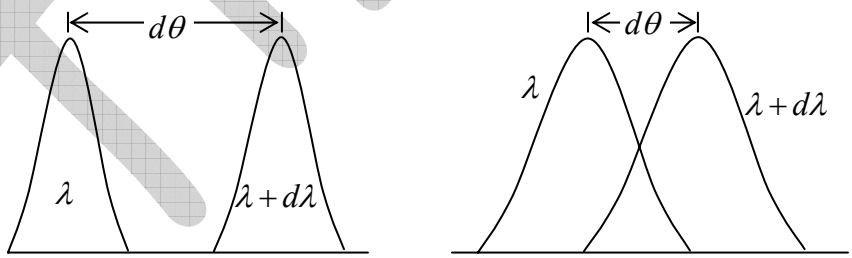
$$\frac{d\theta}{d\lambda} = \frac{n}{(b+e)\cos\theta}$$

Thus, higher is the order n of the spectrum or closer are the rulings on the grating (i.e. smaller the value of $(b+e)$ greater is the dispersive power.

The resolving power of the grating, on the other hand, expresses the degree of closeness which the spectral lines can have and yet be distinguished as two. It is measured by $\frac{\lambda}{d\lambda}$, where $d\lambda$ is the smallest wavelength λ. The value of Resolving Power is given by

$$\frac{\lambda}{d\lambda} = nN$$

where, N is the total number of rulings on the grating and n is order of diffraction. Thus greater is the width of the ruled surface, higher is the resolving power. The higher resolving power results in sharp maxima.



(a) (b)

Resolving power is greater in case (a) than case (b).

Example: When a gaseous element is raised to a very high temperature, the atoms emit radiation having discrete wavelengths. Tow strong components in the atomic spectrum have wavelengths of $5890\,\mathring{A}$ & $5896\,\mathring{A}$.

(a) What resolving power must a grating have if these wavelengths are to be distinguished?

(b) To resolve these lines in the second-order spectrum, how many slits of the grating must be illuminated?

Solution:

(a) Resolving power is defined as

$$R = \frac{\lambda}{\Delta\lambda} \text{ where } \lambda = \frac{\lambda_1 + \lambda_2}{2} = \frac{5890 + 5896}{2} = 5893\,\mathring{A}$$

$$\Delta\lambda = \lambda_2 - \lambda_1 = 5896 - 5890 = 6\,\mathring{A}$$

$$\therefore R = \frac{5893\,\mathring{A}}{6\,\mathring{A}} = 982 \quad \Rightarrow R = 982$$

(b) Resolving power also defined as $R = nN$ where n is the order of spectrum and N is number of slits

$$N = \frac{R}{n} = \frac{982}{2} = 491 \text{ slits}$$

Example: A beam of light is incident normally on a diffraction grating of width 1 *cm*. It is found that at 30°, the nth order diffraction maximum for λ_1 = 600 Å is super-imposed on the $(n+1)^{th}$ order for λ_2 = 500 Å. How many lines per cm does the grating have? Find out whether the first order spectrum from such a grating can be used to resolve the wavelengths λ_3 = 5800 Å and λ_4 = 5802 Å?

Solution:

Condition for diffraction maxima is

$$(b+e)\sin\theta = n\lambda$$

For given two wavelength the maxima condition can be written as

$$(b+e)\sin 30^0 = n\lambda_1 \quad and \quad (b+e)\sin 30^0 = (n+1)\lambda_2$$

Taking ration of the two, we get

$$n = \frac{\lambda_2}{\lambda_1 - \lambda_2} = \frac{5000}{1000} = 5$$

The grating element is

$$(b+e)\sin 30^0 = 5\times 500\,\overset{0}{\text{A}} \Rightarrow b+e = 5\times 10^{-5}\,cm$$

Number of grating is

$$N = \frac{Grating \quad width}{Grating \quad element} = \frac{W}{b+e} = \frac{1}{5\times 10^{-5}} = 20000$$

Resolving power of this Grating is

$$R = \frac{\lambda}{d\lambda} = nN\,.$$

For 1st order spectrum, the resolving power of the given grating is

$$R = nN = 20000$$

Whereas to resolve the wavelengths λ_3 = 5800 Å and λ_4 = 5802 Å the required resolving power is

$$R = \frac{\lambda}{d\lambda} = \frac{5800}{2} = 2900\,.$$

Resolving power of the given grating (20000) is more than required(2900). Thus, given grating will resolve the wavelengths $\lambda_3 = 5800\,\overset{0}{A}$ and $\lambda_4 = 5802\,\overset{0}{A}$.

MCQ (Multiple Choice Questions)

Q1. Consider Fraunhofer diffraction pattern obtained with a single slit illuminated at normal incidence. At the angular position of the first diffraction minimum, the phase difference (in radian) between the wavelets from the opposite edges of the slit is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) 2π (d) π.

Q2. A plane transmission grating has 800 lines and $0.5\ cm$ width. Light consisting of two doublets I and II falls on it normally. The mean wavelengths of doublets I and II are $6000\,\overset{0}{A}$ and $4200\,\overset{0}{A}$, respectively, and the wavelength separation for both is $6\,\overset{0}{A}$. Which one of the following is correct in first-order spectrum ?

(a) Both the doublets are not resolved

(b) Doublet I is not resolved, but doublet II is resolved

(c) Both the doublets are resolved

(d) Doublets I is resolved, but doublet II is not resolved

Q3. In diffraction grating, slit width is b and ruling separation is e. If second order spectrum is to be avoided, one should choose:

(a) $b=\frac{e}{2}$ (b) $b=2e$ (c) $b=\sqrt{2e}$ (d) $b=e$

Q4. A diffraction grating has N lines and grating element is $(b+e)$. For wavelength λ, what is the maximum resolving power possible?

(a) $\frac{N(b+e)}{\lambda}$ (b) $\frac{(b+e)}{(N\lambda)}$

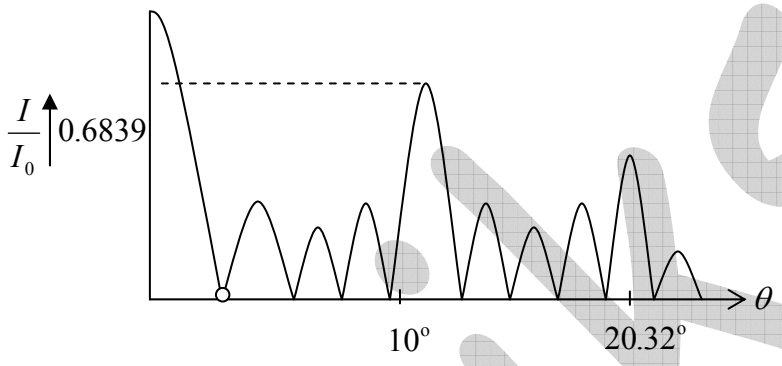
(c) $(b+e)\frac{N^2}{\lambda}$ (d) $(b+e-\lambda)^2\frac{N}{\lambda^2}$

Q5. A $10mW$ laser beam $(\lambda_0 = 0.6\,\mu m)$ is launched on a lens of focal length $10\,cm$. The diameter of the laser beam is $4\,mm$. The area of the focused spot is approximately given by:

(a) $2.25\times10^{-10}\ cm^2$ (b) $2.25\times10^{-6}\ cm^2$

(c) $2.25\times10^{-8}\ cm^2$ (d) $2.25\times10^{-9}\ cm^2$

Q6.



A typical N slit diffraction grating intensity distribution is shown in the figure above. Assume $\lambda = 6\times10^{-5}\ cm$. The value of N given by:

(a) 4 (b) 5 (c) 6 (d) 7

Q7. A screen is placed at a distance D from a narrow diffracting slit of width $b << D$. What is the separation between central diffraction maxima and the first minima, if wavelength of diffraction light λ?

(a) $\frac{Db}{\lambda}$ (b) $\frac{\lambda^2}{D}$ (c) $\frac{D\lambda}{b}$ (d) $\frac{b\lambda}{D}$

Q8. The angular separation between two wavelengths λ and $\lambda + d\lambda$ in a diffraction grating is directly proportional to:

(a) frequency of light (b) grating element

(c) width of grating (d) wavelength of light

Q9. In a plane diffraction grating with N elements, the intensity of the diffracted beam is:

(a) Proportional to $\sqrt{N}$ (b) Proportional to N

(c) Proportional to N^2 (d) independent of N

Q10. Which one of the following plane transmission gratings of width W, and number of lines per centimeter N, will have the maximum resolving power in the first order?

(a) $W = 1\ cm,\ N = 5000$ (b) $W = 1.5\ cm,\ N = 4000$

(c) $W = 2\ cm,\ N = 2400$ (d) $W = 3\ cm,\ N = 1500$

Q11. A grating which would be most suitable for constructing a spectrometer for the visible and ultraviolet region should have:

(a) 100 lines/cm (b) 1000 lines/cm

(c) $10,000$ lines/cm (d) $10,00,000$ lines/cm

Q12. A parallel beam of light of wavelength $600\ nm$ gets diffracted by a single-slit of width $0.2\ mm$. The angular divergence of the first maxima of diffracted light is

(a) $6\times10^{-3}\ rad$ (b) $3\times10^{-3}\ rad$

(c) $4.5\times10^{-2}\ rad$ (d) $9.0\times10^{-2}\ rad$

Q13. A single slit is used to observe diffraction pattern with red light. On replacing the red light with violet light the diffraction pattern would:

(a) remain unchanged (b) become narrower

(c) become broader (d) disappear

Q14. The minimum number of lines in a grating which will just resolve the spectral lines of wavelengths $5880\ \mathring{A}$ and $5886\ \mathring{A}$ in the second order is:

(a) 491 (b) 981 (c) 2940 d) 2943

Q15. In the far-field diffraction pattern of a single slit under polychromatic illumination, the first minima with the wavelength λ_1 is found to be coincident with the third minima at λ_2 than the relationship between these two wavelengths is:

(a) $3\lambda_1 = 0.3\lambda_2$ (b) $3\lambda_1 = \lambda_2$

(c) $\lambda_1 = 3\lambda_2$ d) $0.3\lambda_1 = 3\lambda_2$

MSQ (Multiple Select Questions)

Q16. Fraunhofer diffraction pattern of multiple identical parallel slits is observed. If the number of slits is decreased, keeping the total aperture width and the slit size constant, how are the fringes affected?

(a) The fringes become brighter
(b) The fringes become less bright
(c) The fringes become broader
(d) The fringes become sharper

Q17. Which of the following can be diffracted?

(a) Radio waves
(b) Sound waves
(c) Microwave
(d) X -ray

Q18. Yellow light of intensity I_0 is used in a single slit diffraction experiment with slit width of $0.6\ mm$ if yellow light is replaced by Red light of same intensity I_0 , then the observed pattern will reveal,

(a) that the central maximum is broader
(b) Intensity of central maxima reduces
(c) Intensity of central maxima remain same
(d) First minima appears at higher angle

NAT (Numerical Answer Type)

Q19. A slit of width d is placed in front of a lens of focal length $0.5\ m$ and is illuminated normally with light of wavelength $5.89\times10^{-7}\ m$. The first diffraction minima on either side of the central diffraction maximum are separated by $2\times10^{-3}\ m$. The width d of the slit is _____ $\times10^{-4}\ m$.

Q20. The wave lengths of sodium D lines are $589.59\ um$ and $588.99\ um$ The minimum number of lines that a grating must have in order to resolve these lines in the first order spectrum is _____.

Q21. In a diffraction grating experiment, the grating has 10^{-5} rulings. Then, in the $\lambda = 5000\ \overset{0}{A}$ region of the spectrum and in the second order, the grating can resolve two line with a wavelength difference of _____ $\overset{0}{A}$

Q22. The angular dispersion of two spectral lines from a plane diffraction grating in first order spectrum is 3° the dispersion in the second order spectrum for the same spectral lines will be approximately _____ degree.

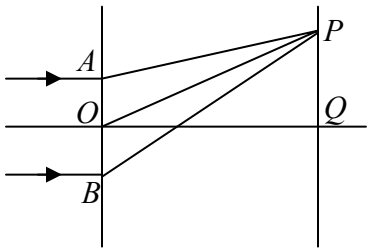
Q23. In a single-sit Fraunhofer diffraction set-up used with light of wavelength $4000\ \overset{0}{A}$, the distance D between the central maximum and first minimum is found to be $0.3\ cm$. In the same setup, if the wavelength of light used is changed to $6000\ \overset{0}{A}$, the corresponding value of D will be _____ cm.

Solution-MCQ (Multiple Choice Questions)

Ans. 1: (c)

Solution: The figure depicts Fraunhofer diffraction pattern.

For P to be first minimum, the phase difference between OA or OB is π.



$\therefore$ Phase difference from opposite edges $= 2\pi$ radian

Ans. 2: (b)

Solution: Resolving power of grating is given as $\frac{\lambda}{d\lambda} = nN$

where $n =$ order of spectrum, $N =$ number of lines in grating

For first order $n = 1$, $\frac{\lambda}{d\lambda} = N$

(i) when $\lambda = 6000\,\overset{0}{A}$, $d\lambda = 6\,\overset{0}{A}$ then $N = \frac{\lambda}{d\lambda} = \frac{6000}{6} = 1000$

Thus, the number of lines required lines on grating is 1000 but actual was 800, so it cannot be resolved.

(ii) when $\lambda = 4200\,\overset{0}{A}$, $d\lambda = 6\,\overset{0}{A}$ $\Rightarrow N = \frac{4200}{6} = 700$

Thus, number of required lines on grating is 700 but actual line is 800. It is resolved.

Ans. 3: (d)

Solution: If slit width is e and ruling separation is d, then the direction of interference maxima are given as $(b+e)\sin\theta = n\lambda$ ….. (i)

The direction minima are given as $b\sin\theta = m\lambda$ ….. (ii)

In equation (i) and (ii) n and m are integers. If the value of b and e are such that the both equations are satisfied simultaneously for the same value of θ, then the position of certain interference maxima correspond to the diffraction minima at the same position on the screen.

For second order missing $\frac{n}{m} = 2 \Rightarrow \frac{b+e}{b} = 2 \Rightarrow b+e = 2b \Rightarrow b = e$

Ans. 4: (a)

Solution: The direction of interference maxima for a grating is given as

$(b+e)\sin\theta = n\lambda \Rightarrow n = \frac{(b+e)\sin\theta}{\lambda}$

Resolving power of grating $R = nN$

n is order of diffraction , N is total number of ruling

From equation (i) it is clear that maximum resolution of diffraction grating occurs at $\theta = 90^0, \sin\theta = 1$

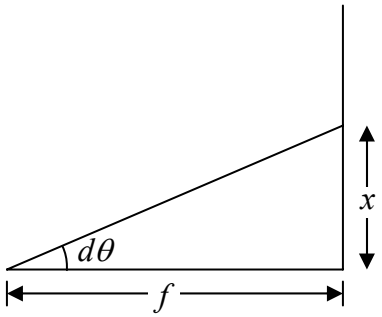
Maximum resolution $= \frac{(b+e)}{\lambda}N$

Ans. 5: (d)

Solution: If D is diameter of the laser beam, then $D\sin\theta = \lambda$

If θ is very small, then $\sin\theta = d\theta$

$Dd\theta = \lambda \Rightarrow \frac{xD}{f} = \lambda \Rightarrow x = \frac{f\lambda}{D}$

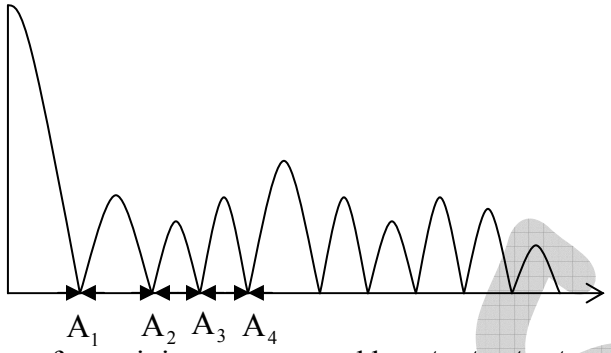


So, area

$\pi x^2 = \pi\left(\frac{f\lambda}{D}\right)^2 = 3.14\times\left(\frac{0.6\times10^{-6}\times10^{-11}}{4\times10^{-3}}\right)^2 \text{ cm}^2 = 3 \times 10^{-19} \text{ cm}^2$

Ans. 6: (b)

Solution: Between two principal maxima of grating have N slit there is $(N-1)$ minima.



From the figure there are four minima represented by A_1, A_2, A_3, A_4

Hence, $N-1=4 \quad \Rightarrow N=5$

Ans. 7: (c)

Solution: If b is width of narrow slit then minima of diffraction pattern is given as

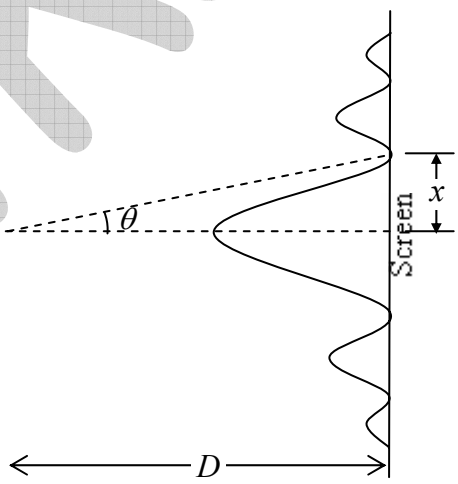
$$b\sin\theta = m\lambda$$

For first minima $m=1$, So, $b\sin\theta=\lambda$

If θ is very small $\sin\theta \cong \tan\theta \cong \frac{x}{D}$

Thus, $\frac{bx}{D}=\lambda \Rightarrow x=\left(\frac{\lambda D}{b}\right)$

$\Rightarrow$ Separation between first minima and central maxima is $\frac{\lambda D}{b}$



Ans. 8: (d)

Solution: The diffraction pattern for a light of wavelength λ and stiff width e separated by a distance b is given as

$$(b+e)\sin\theta_n = n\lambda \Rightarrow (b+e)\cos\theta_n . d\theta_n = nd\lambda$$

$$\Rightarrow \quad d\theta_n = \frac{nd\lambda}{(b+e)\cos\theta_n}$$

Ans. 9: (c)

Solution: In plane grating the intensity is proportion to square of number of lines drawn on the grating $I \propto N^2$

Ans. 10: (b)

Solution: The resolving power of a grating is given as resolving power $= nN$

Where n is order and N is total number of line of grating. Thus, resolving power of a grating is proportional to the total number of liens on it.

(a) total number of lines $= W \times N = 1 \times 5000 = 5000$

(b) total number of lines $= 1.5 \times 4000 = 6000$

(c) total number of lines $= 2 \times 2400 = 4800$

(d) total number of lines $= 3 \times 1500 = 4500$

Hence, total numbers of lines are maximum for option (b).

Ans. 11: (c)

Solution: In a grating the maxima is given as $(b+e)\sin\theta = n\lambda \Rightarrow \frac{n\lambda}{b+e} = \sin\theta \le 1$

$\Rightarrow \frac{n}{b+e} \le \frac{1}{\lambda} \Rightarrow nN \le \frac{1}{\lambda}$ where $N = \frac{1}{b+e}$ is number of lines per centimeter (cm) for visible and ultraviolet region we can take λ as $1000\,\overset{0}{A}$ and n as 10

$$N \le \frac{1}{1000 \times 10^{-8} \times 10} \Rightarrow N \le 10,000$$

Ans. 12: (b)

Solution: If a is width of a slit the diffraction pattern is given as

$$b\sin\theta = n\lambda \Rightarrow \sin\theta = \frac{n\lambda}{b}$$

If θ is very small then $\theta = \frac{n\lambda}{b}$

Here, $n = 1, \lambda = 600 \times 10^{-9}\ m,\ b = 0.2 \times 10^{-3}$

So, $\theta = \frac{1 \times 600 \times 10^{-9}}{0.2 \times 10^{-3}} = 3 \times 10^{-3}\ rad$

Ans. 13: (b)

Solution: The diffraction pattern is given as

$$b\sin\theta = n\lambda$$

Since the wavelength of violet light is less than that of red light so if red light is replaced by violet light the diffraction pattern becomes narrower.

Ans. 14: (a)

Solution: The resolving power of grating is given $\frac{\lambda}{\Delta\lambda} = nN$

Here, $n = 2$ So, $N = \frac{\lambda}{2\Delta\lambda} \Rightarrow N = \frac{5880}{2\times 6} = 491$ $\because \Delta\lambda = 5886 - 5880 = 6$

Ans. 15: (c)

Solution: Minima $= n\lambda$

$\Rightarrow$ $n_1\lambda_1 = n_2\lambda_2 \Rightarrow \lambda_1 = 3\lambda_2$

MSQ (Multiple Select Questions)

Ans. 16: (b) and (d)

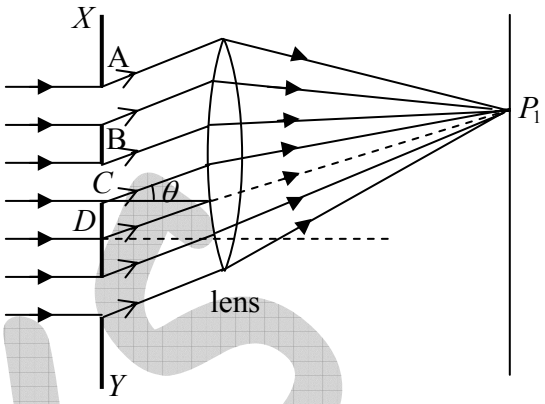
Solution: Let the width of each slit is a and b is opaque spacing between two consecutive slits.

Then the nth order principal maxima is given as

$$(b+e)\sin\theta = n\lambda$$

If the number of slits is decreased keeping the other terms constant then fringes becomes

(i) less brighter and (ii) broader



Ans. 17: (a), (b), (c) and (d)

Solution: The diffraction can take place in all above waves.

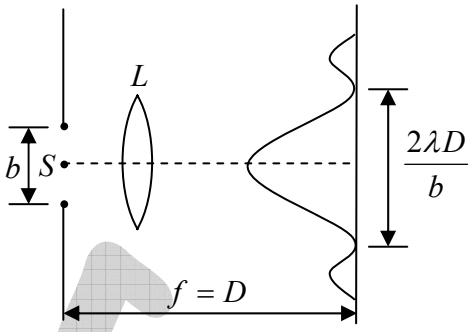
Ans. 18: (a), (c) and (d)

Solution: Since λ (Red-light) is more than yellow light. Thus the fist minima appears at higher angle as a result the width of the central maxima increases. But intensity of central maxima remain same as incident intensity is unchanged.

NAT (Numerical Answer Type)

Ans. 19: 2.945

Solution: width of central maxima $= \dfrac{2\lambda D}{b}$, during diffraction.



or $2\times10^{-3} = \dfrac{2\times\left(5.89\times10^{-7}\right)\times0.5}{b}$

or $b = 2.945\times10^{-4}\ m$.

Ans. 20: 988

Solution: Here, $\lambda_1 = 589.59\times10^{-6}\ m$, $\lambda_2 = 599.59\times10^{-6}\ m$

so, $\Delta\lambda = \lambda_1 - \lambda_2 = 589.59\times10^{-6} - 588.99\times10^{-6} = 0.597\times10^{-6}\ m$

The resolving power of grating is given as $\dfrac{\lambda}{\Delta\lambda} = nN$

where n is the order of diffraction and N is the number of slits in grating.

Here $n = 1$

By Equation (i) and (ii) $N = \dfrac{\lambda}{\Delta\lambda n} = \dfrac{589.59\times10^{-6}}{0.597\times10^{-6}}$

Therefore, the minimum number of lines required for just resolution in the first order in 988.

Ans. 21: 0.025

Solution: The grating can resolve, if $\dfrac{\lambda}{d\lambda} = nN$

Here $\lambda = 5000\ \mathring{A} \Rightarrow \lambda = 5000\times10^{-10} m$, $n = 2$ and $N = 105$

So, $\dfrac{1}{d\lambda} = \dfrac{nN}{\lambda} \Rightarrow d\lambda = \dfrac{\lambda}{nN} = \dfrac{5000\times10^{-10}}{2\times10^{5}}\ \mathring{A} = 0.025\mathring{A}$

Ans. 22: 6°

Solution: **Dispersive power of a grating:** The dispersive power of a grating is defined as the ratio of difference in angle of diffraction of any two neighbouring spectral lines to the difference in wavelength between two spectral lines. It can also be defined as difference in angle of diffraction per unit change in wavelength.

The diffraction of the nth order principal maxima for a wavelength λ is given as

$$(b+e)\sin\theta = n\lambda$$

Differentiating $(b+e)\cos\theta.d\theta = nd\lambda \Rightarrow \frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta} \Rightarrow \frac{d\theta}{d\lambda} \propto n$

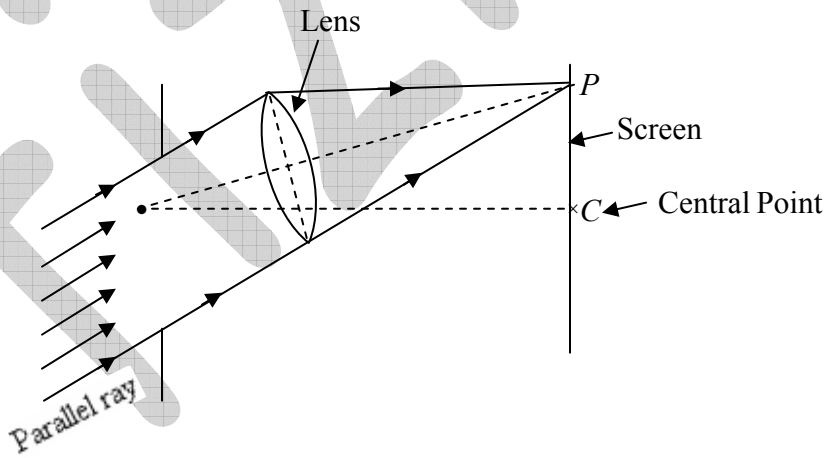
Thus, dispersive power $\propto n$

Here, $\left(\frac{d\theta}{d\lambda}\right) = An$ $\quad A = \text{constant}$

$\left(\frac{d\theta}{d\lambda}\right)_1 = 3^{o}, n = 1$ and $\left(\frac{d\theta}{d\lambda}\right)_2 = \left(\frac{d\theta}{d\lambda}\right)\times\left(\frac{n_2}{n_1}\right) 3^{o}\times\frac{2}{1} = 6^{o}$

Ans. 23: 0.45

Solution:



Fraunhofer diffraction due to single slit. If b is diameter of lens the minima for diffraction pattern is given as $b\sin\theta = n\lambda$

$\Rightarrow$ $\quad (\text{distancce}) \propto \lambda$

$\Rightarrow$ $\quad \text{distance} = (0.3\ cm)\times\left(\frac{6000}{4000}\right) = 0.45\ cm$

4. Polarization of Light

Plane of Incidence

Incident ray, reflected ray, refracted ray and normal to incidence forms a plane that plane is plane of incidence.

Plane of Polarization

The plane passing through the direction of propagation and containing no vibration is called the "Plane of Polarization".

4.1 Production of Plane Polarized Light

Different methods of production of polarized light

(i) Polarization by reflection

(ii) Polarization by refraction

(iii) Polarization by selective absorption

(iv) Polarization by double refraction

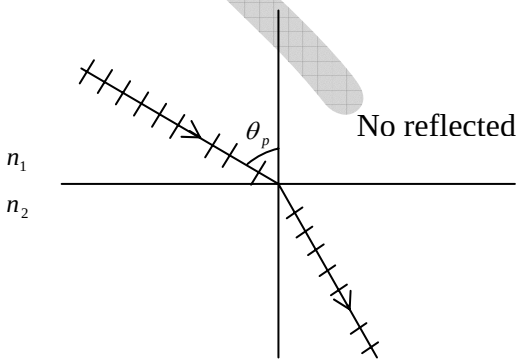
(v) Polarization by scattering

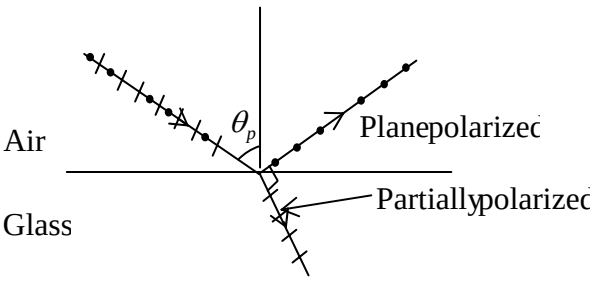
4.1.1 Polarization by Reflection

If a linearly polarized wave (Electric vector associated with the incident wave lies in the plane of incidence) is incident on the interface of two dielectrics with the angle of incidence equal to θ. If the angle of incidence θ is such that

$$\theta = \theta_p = \tan^{-1}\left(\frac{n_2}{n_1}\right)$$

then the reflection coefficient is zero.





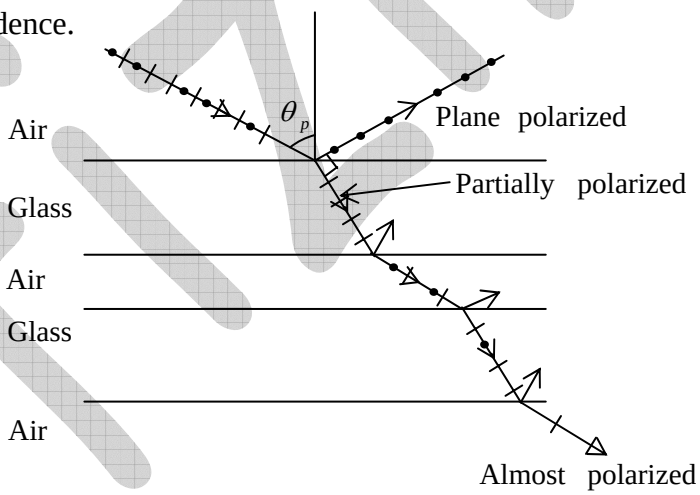
Thus if an unpolarized beam is incident with an angle of incidence equal to θ_p, the reflected beam is plane polarized whose electric vector is perpendicular to the plane of incidence.

Above equation is known as Brewster's law. The angle θ_p is known as the polarizing angle or the Brewster angle. At this angle, the reflected and the refracted rays are at right angle to each other i.e. $\theta_p + r = \frac{\pi}{2} \Rightarrow n = \frac{\sin\theta_p}{\sin r} = \frac{\sin\theta_p}{\cos\theta_p} \Rightarrow \tan\theta_p = n$

For the air-glass interface, $n_1 = 1$ and $n_2 \approx 1.5$ giving $\theta_p \approx 57^0$.

4.1.2 Polarization by Refraction

If an unpolarized beam is incident with an angle of incidence equal to θ_p, the reflected beam is plane polarized whose electric vector is perpendicular to the plane of incidence. The transmitted beam is partially polarized and if this beam is made to undergo several reflections, then the emergent beam is almost plane polarized with its electric vector in the plane of incidence.

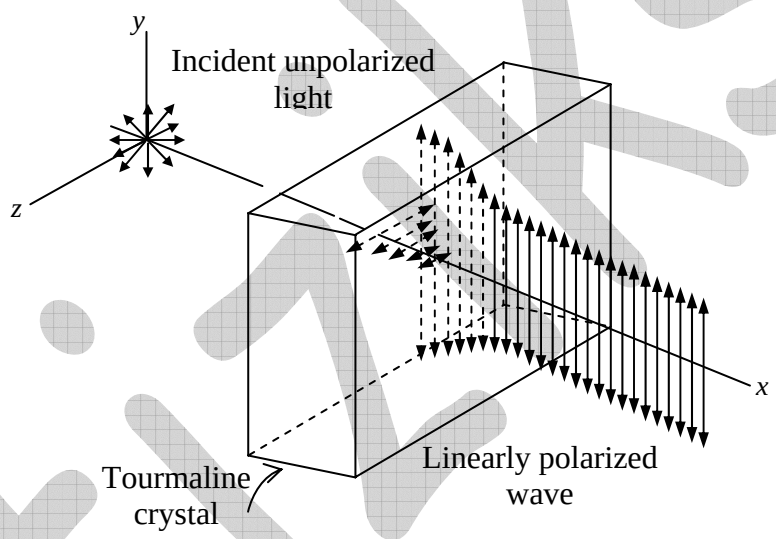


If I_p and I_s be the intensity of the parallel and perpendicular component in refracted light, then the degree of polarization is given by $P = \frac{I_p - I_s}{I_p + I_s} = \frac{m}{m + \left(\frac{2n}{1-n^2}\right)^2}$

where m is the number of plate and n is the refractive index.

4.1.3 Polarization by selective absorption

A simple method for eliminating one of the beams is through selective absorption; this property of selective absorption is known as dichroism. A crystal such as tourmaline has different coefficients of absorption for the two linearly polarized beams into which the incident beam splits up. Consequently, one of the beams gets absorbed quickly, and the other component passes through without much attenuation. Thus, if an unpolarized beam is passed through a tourmaline crystal, the emergent beam will be linearly polarized

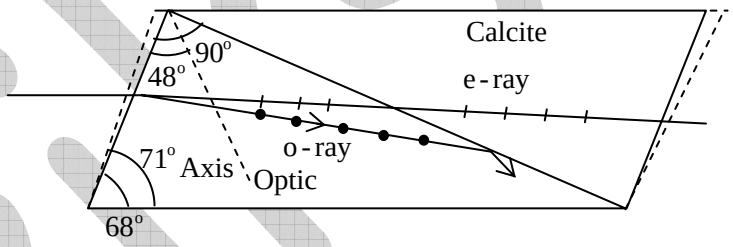


4.1.4 Polarization by Double Refraction

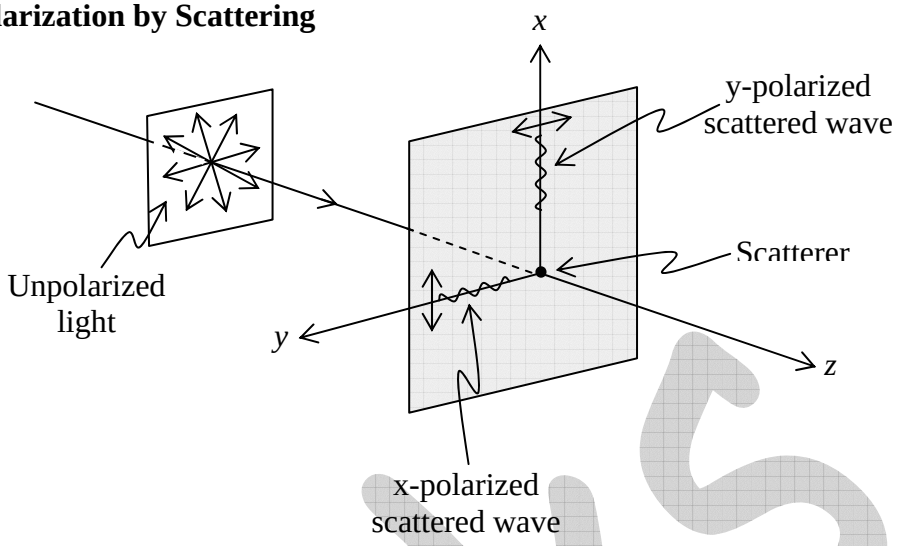
When a ray of unpolarized light passed through doubly refracting crystal (calcite or quartz), it split up into two refracted rays. One of the ray obeys the ordinary laws of refraction i.e. (n remains constant) and it is called 'ordinary ray' (o-ray). The other behaves in an extra ordinary way (i.e. n varies) and called 'extraordinary ray' (e-ray).

It is found that both the ordinary and extraordinary rays are plane-polarized having vibration perpendicular to each other.

If one can sandwich a layer of a material whose refractive index lies between the two, then for one of the beams, the incidence will be at a rarer medium and for the other it will be at a denser medium. This principle is used in a Nicol prism which consists of a calcite crystal cut in such a way that for the beam, for which the sandwiched material is a rarer medium, the angle of incidence is greater than the critical angle. Thus this particular beam will be eliminated by total internal reflection. Following figure shows a properly cut calcite crystal in which a layer of Canada balsam has been introduced so that the ordinary ray undergoes total internal reflection. The extraordinary component passes through, and the beam emerging from the crystal is linearly polarized.



4.1.5 Polarization by Scattering



When an unpolarized beam moving in z-axis, strike a gas atom than the electric field in unpolarized beam sets the electric charges in the gas atom in vibration. Since the E-field is in xy plane, therefore, the vibration will take place in xy plane only. These vibration can resolve in x and y direction and can said that two dipoles oscillating one in x and other in y-direction with the frequency of incident light. Since an oscillating dipole does not radiate in the direction of its own length.

Therefore, observer along x-axis will receive light component vibrating only in y-direction and along y-axis the component will vibrate only x axis. Hence observer will receive Plane Polarized light.

Blue colour of sky

Lord Rayleigh Formula: Intensity of light scattered from firie particles (dimension $\leq \lambda$) varies inversely as the fourth power of λ. i.e. $I \propto \frac{1}{\lambda^4}$

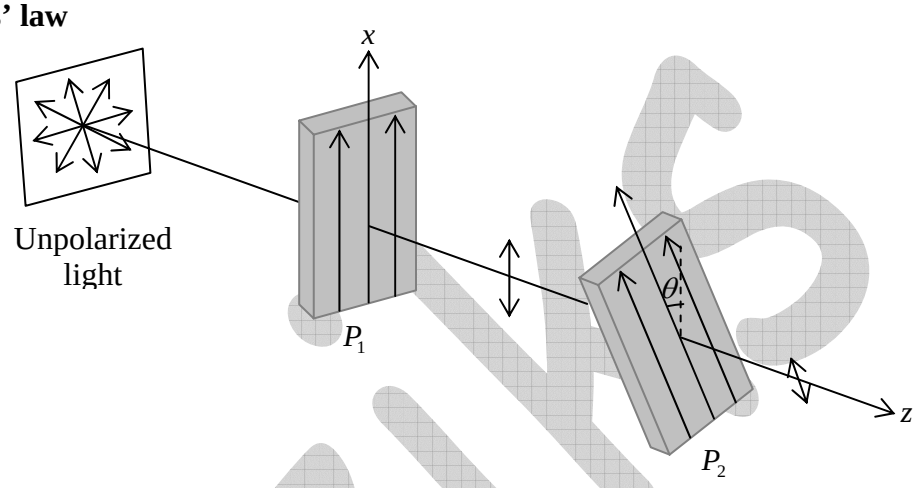
Scattered intensity of blue light is more than red $\because \lambda_{blue} < \lambda_{red}$.

Why blue scattered more: Natural frequency of bound electron in gas lies in ultra-violet region. Blue light closer to natural frequency than red light, therefore, blue is more effective in causing the electron to oscillate and thus it get more effectively scattered than red.

Red colour of sunrise and sunset

The path of the light through the atmosphere at sunrise and sunset is greatest. Since the violet & blue light is largely scattered and get removed and what we see is rest having red component.

4.2 Malus' law



An unpolarized light beam gets polarized after passing through the Polaroid P_1 which has a pass axis parallel to the x axis. When this x-polarized light beam incident on the second Polaroid P_2 whose pass axis makes an angle θ with the x axis, than the intensity of the emerging beam will vary as

$$I = I_0 \cos^2 \theta$$

where I_0 represents the intensity of the emergent beam when the pass axis of P_2 is also along the x axis (i.e., when $\theta = 0$), above equation known as Malus' law.

Thus, if a linearly polarized beam is incident on a Polaroid and if the Polaroid is rotated about the z axis, then the intensity of the emergent wave will vary according to the above law.

4.3 Superposition of Two Disturbances and Production of Polarized Wave

4.3.1 Superposition of Two Waves with Parallel Electric Field

Let us consider the propagation of two linearly polarized electromagnetic waves (both propagating along the z axis) with their electric vectors oscillating along the x axis. The electric fields associated with the waves can be written in the form

$$E_1 = \hat{x}a_1 \cos\left(kz - \omega t + \theta_1\right) \tag{1}$$

$$E_2 = \hat{x}a_2 \cos\left(kz - \omega t + \theta_2\right) \tag{2}$$

where a_1 and a_2 represent the amplitudes of the waves, $\hat{x}$ represents the unit vector along the x axis, and θ_1 and θ_2 are phase constants. The resultant of these two waves is given by

$$E = E_1 + E_2 \tag{3}$$

which can always be written in the form

$$E = \hat{x}a\cos\left(kz - \omega t + \theta\right) \tag{4}$$

where $$a = \left[a_1^2 + a_2^2 + 2a_1a_2\cos\left(\theta_1 - \theta_2\right)\right]^{\frac{1}{2}} \tag{5}$$

represents the amplitude of the wave. Equation (4) tells us that the resultant is also a linearly polarized wave with its electric vector oscillating along the same axis.

4.3.2 Superposition of Two Waves with Mutually Perpendicular Electric field

We next consider the superposition of two linearly polarized electromagnetic waves (both propagating along the z axis) but with their electric vectors oscillating along two mutually perpendicular directions. Thus, we may have

$$E_1 = \hat{x}a_1 \cos\left(kz - \omega t\right) \tag{6}$$

$$E_2 = \hat{y}a_2 \cos\left(kz - \omega t + \theta\right) \tag{7}$$

For $\theta = n\pi$, the resultant will also be a linearly polarized wave with its electric vector oscillating along a direction making a certain angle with the x axis; this angle will depend on the relative values of a_1 and a_2.

To find the state of polarization of the resultant field, we consider the time variation of the resultant electric field at an arbitrary plane perpendicular to the z axis which we may, without any loss of generality, assume to be $z = 0$.

If E_x and E_y represent the x and y components of the resultant field $E = (E_1 + E_2)$, then

$$E_x = a_1 \cos \omega t \tag{8}$$

and

$$E_y = a_2 \cos(\omega t - \theta) \tag{9}$$

where we have used Equations (6) and (7) with $z = 0$.

For $\theta = n\pi$ the above equations simplify to

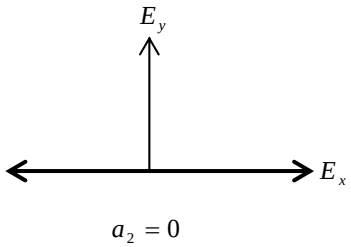
$$E_x = a_1 \cos \omega t \quad \text{and} \quad E_y = (-1)^n a_2 \cos \omega t \tag{10}$$

from which we obtain $\dfrac{Ey}{E_x} = \pm \dfrac{a_2}{a_1}$ (independent of t) (11)

where the upper and lower signs correspond to n even and n odd, respectively. In the $E_x E_y$ plane, Eq. (11) represents a straight line; the angle ϕ that this line makes with the E_x axis depends on the ratio $\dfrac{a_2}{a_1}$. In fact $\phi = \tan^{-1}\left(\pm \dfrac{a_2}{a_1}\right)$ (12)

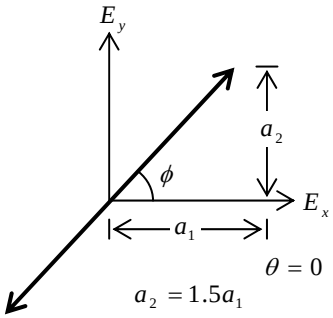
The condition $\theta = n\pi$ implies that the two vibrations are either in phase $(n = 0, 2, 4....)$ or out of phase $(n = 1, 3, 5, ...)$. Thus, the superposition of two linearly polarized electromagnetic waves with their electric fields at right angles to each other and oscillating in phase is again a linearly polarized wave with its electric vector, in general, oscillating in a direction which is different from the fields of either of the two waves.

Following figures shows the plot of the resultant field corresponding to Eq. (10) for various values of $\dfrac{a_2}{a_1}$. The tip of the electric vector oscillates (with angular frequency ω) along the thick lines shown in the figure. The equation of the straight line is given by Eq. (11).



$a_2 = 0$

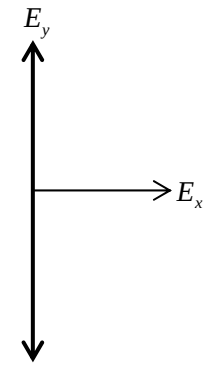
(i)



$\theta = 0$

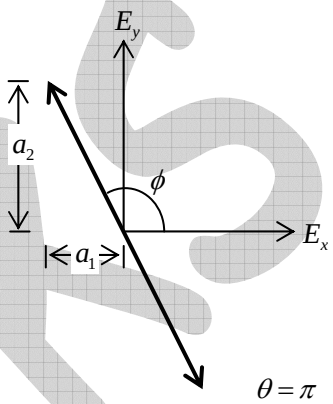
$a_2 = 1.5a_1$

(ii)



$a_1 = 0$

(iii)



$\theta = \pi$

$a_2 = 1.5a_1$

(iv)

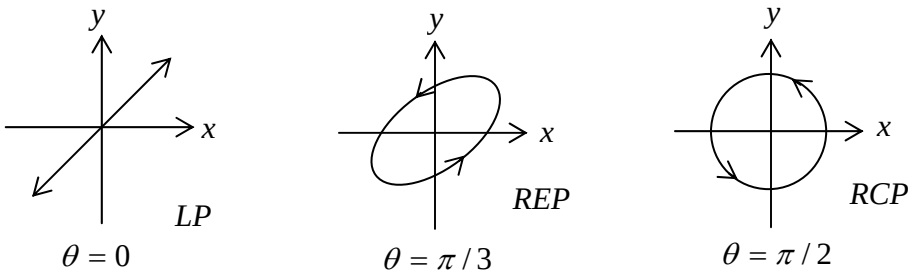
For $\theta \neq n\pi\,(n = 0,1,2,...)$, the resultant electric vector does not, in general, oscillate along a straight line.

We first consider the simple case corresponding to $\theta = \dfrac{\pi}{2}$ with $a_1 = a_2$. Thus,

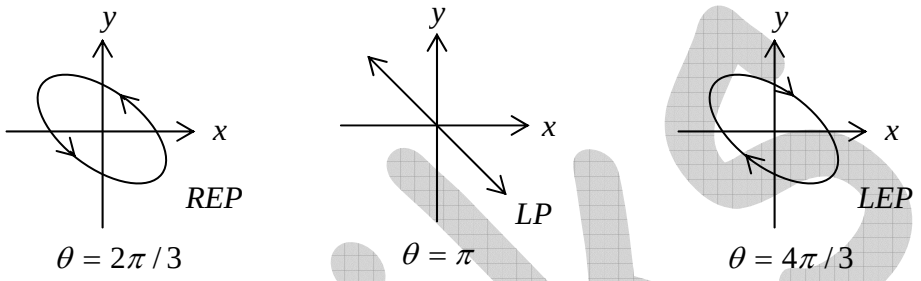
$E_x = a_1 \cos \omega t$ (13) and $E_y = a_1 \cos \omega t$ (14)

If we plot the time variation of the resultant electric vectors whose x and y components are given by Eqs. (13) and (14), we find that the tip of the electric vector rotates on the circumference of a circle (of radius a_1) in the counterclockwise direction as shown in Fig. (c) below, and the propagation is in the $+z$ direction which is coming out of the page. Such a wave is known as a right circularly polarized wave (usually abbreviated as a RCP wave). That the tip of the resultant electric vector should lie on the circumference of circle is also obvious from the fact that

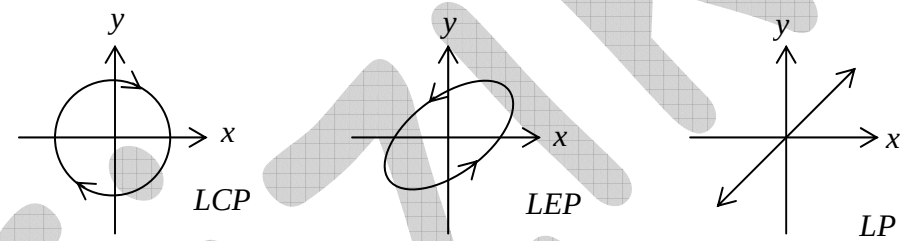
$$E_x^2 + E_y^2 = a_1^2 \qquad \text{(Independent of } t\text{)}$$



$\theta = 0$ (a) $\theta = \pi/3$ (b) $\theta = \pi/2$ (c)



$\theta = 2\pi/3$ (d) $\theta = \pi$ (e) $\theta = 4\pi/3$ (f)



$\theta = 3\pi/2$ (g) $\theta = 5\pi/3$ (h) $\theta = 2\pi$ (i)

For $\theta = \frac{3\pi}{2}$, $E_x = a_1 \cos \omega t$ (15) and $E_y = -a_1 \sin \omega t$ (16)

which would also represent a circularly polarized wave; however, the electric vector will rotate in the clockwise direction [Fig. (g)]. Such a wave is known as a left circularly polarized wave (usually abbreviated as a LCP wave). For $\theta \neq \frac{m\pi}{2}(m = 0,1,2,...)$, the tip of the electric vector rotates on the circumference of an ellipse. As can be seen from the figure, this ellipse will degenerate into a straight line or a circle when θ becomes an even or an odd multiple of $\frac{\pi}{2}$. In general, when $a_1 \neq a_2$, one obtains an elliptically polarized wave which degenerates into a straight line for $\theta = 0, \pi, 2\pi, ...$ etc.

4.4 The Phenomenon of Double Refraction

When an unpolarized light beam is incident normally on a calcite crystal, it would in general, split up into two linearly polarized beams as shown in Fig. (a). The beam which travels undeviated is known as the ordinary ray (usually abbreviated as the o – ray) and obeys Snell's laws of refraction. On the other hand, the second beam, which in general does not obey Snell's laws, is known as the extraordinary ray (usually abbreviated as the e – ray).

The appearance of two beams is due to the phenomenon of double refraction, and a crystal such as calcite is usually referred to as a double refracting crystal. If we put a Polaroid PP' behind the calcite crystal and rotate the Polaroid (about NN'), then for two positions of the Polaroid (when the pass axis is perpendicular to the plane of the paper) the e - ray will be completely blocked and only the o-ray will pass through.

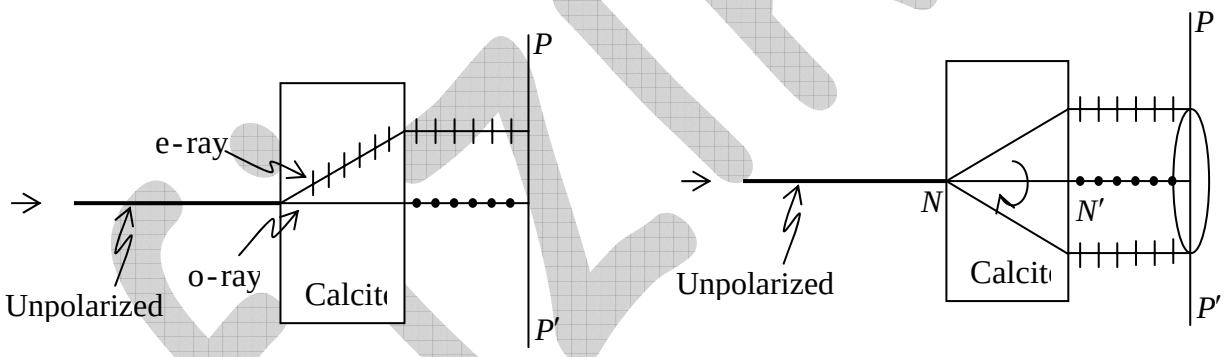


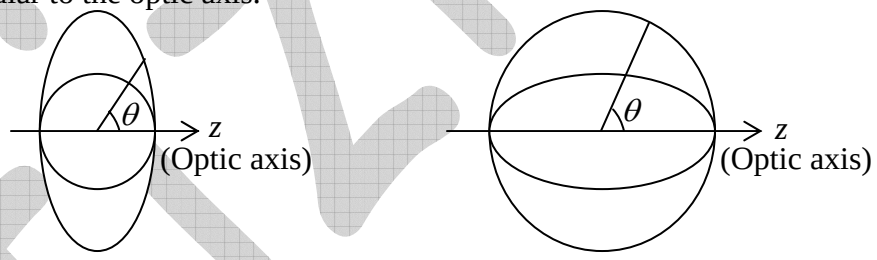
Fig (a) When an unpolarized light beam is incident normally on a calcite crystal, it would in general, split up into two linearly polarized beams. (b) If we rotate the crystal about NN' then the e-ray will rotate about NN'.

On the other hand, when the pass axis of the Polaroid is in the plane of the paper (i.e., along the line PP'), then the o - ray will be completely blocked and only the e-ray will pass through. Further, if we rotate the crystal about NN' then the e -ray will rotate about the axis [see Fig. (b)].

The velocity of the ordinary ray is the same in all directions, the velocity of the extraordinary ray is different in different directions; a substance (such as as calcite, quartz) which exhibits different properties in different directions is called an anisotropic substance. Along a particular direction (fixed in the crystal), the two velocities are equal; this direction is known as the optic axis of the crystal. In a crystal such as calcite, the two rays have the same speed only along one direction (which is the optic axis); such crystals are known as uniaxial crystals. The velocities of the ordinary and the extraordinary rays are given by the following equations:

$$v_{ro} = \frac{c}{n_o} \text{ (Ordinary ray)} \quad \text{and} \quad \frac{1}{v_{re}^2} = \frac{\sin^2\theta}{\left(\frac{c}{n_e}\right)^2} + \frac{\cos^2\theta}{\left(\frac{c}{n_o}\right)^2} \text{ (extraordinary)}$$

where n_o and n_e are refractive index for O - ray and e - ray and θ is the angle that the ray makes with the optic axis; we have assumed the optic axis to be parallel to the z axis. Thus, $\frac{c}{n_o}$ and $\frac{c}{n_e}$ are the velocities of the extraordinary ray when it propagates parallel and perpendicular to the optic axis.



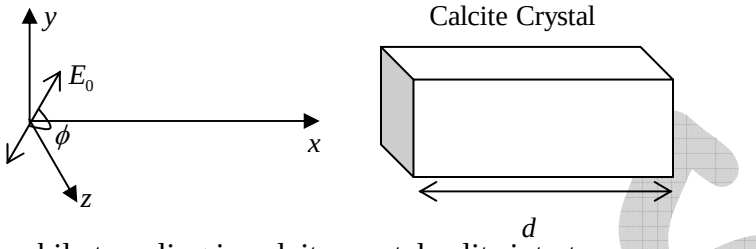
Negative crystal (Calcite) (a)

Positive crystal (Quartz) (b)

Fig. (a) In a negative crystal, the ellipsoid of revolution (which corresponds to the extra ordinary ray) lies outside the sphere; the sphere corresponds to the ordinary ray. (b) In a positive crystal, the ellipsoid of revolution (which corresponds to the extraordinary ray) lies inside the sphere.

4.5 Quarter Wave Plate and Half Wave Plate

Let electric field vector (of amplitude E_0) associated with the incident linearly polarized beam makes an angle ϕ with optic axis which is parallel to *z*-axis and incident on calcite crystal of thickness d whose optic axis is parallel to the surface



Such beam while traveling in calcite crystal splits into two components. The *z*-axis whose amplitude is $E_0\cos\phi$ passes through as an extraordinary ray (e-ray) propagates with velocity c/n_e. The *y*-axis whose amplitude is $E_0\sin\phi$ passes through as an ordinary ray (o-ray) propagates with velocity c/n_o.

Since $n_o \neq n_e$ the two beams will propagate with different velocities, thus when they come out of the crystal, they will not be in phase. Let on the plane $x=0$, the beam is incident then

$$E_y = E_0\sin\phi\cos(kx-\omega t) \quad \text{and} \quad E_z = E_0\cos\phi\cos(kx-\omega t).$$

Thus at $x=0$, we have $E_y = E_0\sin\phi\cos\omega t$ and $E_z = E_0\cos\phi\cos\omega t$.

Inside crystal the two components will be

$$E_y = E_0\sin\phi\cos(n_o kx-\omega t) \quad \text{and} \quad E_z = E_0\cos\phi\cos(n_e kx-\omega t).$$

If thickness of the crystal is d then on emerging surface we have

$$E_y = E_0\sin\phi\cos(n_o kd-\omega t) \quad \text{and} \quad E_z = E_0\cos\phi\cos(n_e kd-\omega t).$$

Thus the phase difference between e-ray and o-ray is $\delta = kd(n_o-n_e)$.

If phase difference is $\delta=\pi/2$, then $d=\dfrac{\lambda}{4(n_o-n_e)}$ (Quarter Wave Plate)

If phase difference is $\delta=\pi$, then $d=\dfrac{\lambda}{2(n_o-n_e)}$ (Half Wave Plate)

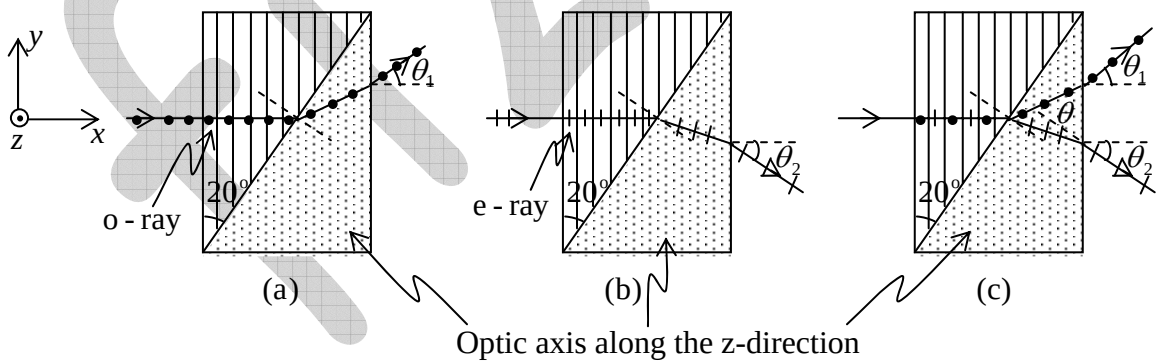
4.6 Wollaston Prism

A Wollaston prism is used to produce two linearly polarized beams. It consists of two similar prisms (calcite) with the optic axis of the first prism parallel to the surface and the optic axis of the second prism parallel to the edge of the prism as shown below. Let us first consider the incidence of a z polarized beam as shown in Fig. (a). The beam will propagate as an $o-$ ray in the first prism (because the vibrations are perpendicular to the optic axis) and will see the refractive index n_0. When this beam enters the second prism, it will become an e-ray and will see the refractive index n_e. For calcite $n_o > n_e$ and the ray will bend away from the normal. Since the optic axis is normal to the plane of paper, the refracted ray will obey Snell's laws, and the angle of refraction will be given by

$$n_0 \sin 20^o = n_e \sin r_1$$

where we have assumed the angle of the prism to be 20°. Assuming $n_o \approx 1.658$ and $n_e \approx 1.486$, we readily get $r_1 \approx 22.43^o$

Thus the angle of incidence at the second surface is $i_1 = 22.43^o - 20^o = 2.43^o$. The output angle θ_1 is given by $n_e \sin 2.43^o = \sin\theta_1 \Rightarrow \theta_1 = 3.61^o$.



We next consider the incidence of a y - polarized beam as shown in Fig. (b). The beam will propagate as an e-ray in the first prism and as an o-ray in the second prism. The angle of refraction is now given by

$$n_e \sin 20 = n_o \sin r_2 \Rightarrow r_2 \approx 17.85^o$$

Thus the angle of incidence at the second interface is

$$i_2 = 20^o - 17.85^o = 2.15^o$$

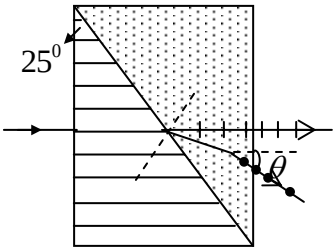
The output angle θ_2 is given by

$$n_o \sin 2.15^o = \sin\theta_2 \Rightarrow \theta_2 \approx 3.57^o$$

Thus, if an unpolarized beam is incident on the Wollaston prism, the angular separation between the two orthogonally polarized beams is $\theta = \theta_1 + \theta_2 \approx 7.18^o$.

4.7 Rochon Prism

We next consider the Rochon prism which consists of two similar prisms of (say) calcite; the optic axis of the first prism is normal to the face of the prism while the optic axis of the second prism is parallel to the edge as shown in figure. Now, in the first prism both beams will see the same refractive index n_o; this follows from the fact that the ordinary and extraordinary waves travel with the same velocity $\left(\frac{c}{n_o}\right)$ along the optic axis of the crystal. When the beam enters the second crystal, the ordinary ray (whose D is normal to the optic axis) will see the same refractive index and go undeviated as shown in figure. On the other hand, the extraordinary ray (whose D is along the optic axis) will see the refractive index n_e and will bend away from the normal. We assume the angle of the prism to be 25°. The angle of refraction will be determined from

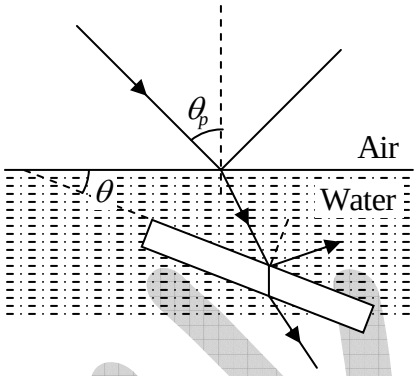


$$n_o \sin 25^o = n_e \sin r$$

Thus $\sin r = \frac{n_o}{n_e}\sin 25^o = \frac{1.658}{1.486} \times 0.423 \approx 0.472 \Rightarrow r = 28.2^o$

Therefore the angle of incidence at the second surface will be $28.2^o - 25^o = 3.2^o$. The emerging angle will be given by $\sin\theta = n_e \sin\left(3.2^o\right) \Rightarrow \theta \approx 4.8^o$

Example: Light strikes a water surface at the polarizing angle. The part of the beam refracted into the water strikes a submerged glass slab (index of refraction,1.50), as shown in Figure. The light reflected from the upper surface of the slab is completely polarized. Find the angle between the water surface and the glass slab

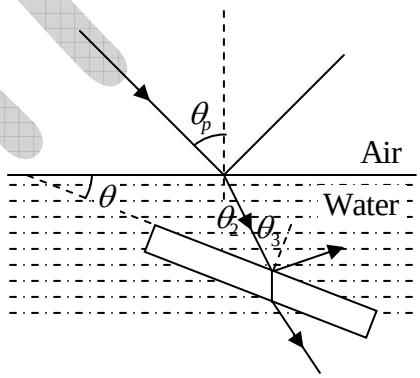


Solution: For the air-to-water interface,

$$\tan\theta_p = \frac{n_{\text{water}}}{n_{\text{air}}} = \frac{1.33}{1.00} \qquad \left(\theta_p = 53.1^{\text{o}}\right)$$

and $(1.00)\sin\theta_p = (1.33)\sin\theta_2$

$$\theta_2 = \sin^{-1}\left(\frac{\sin 53.1^{\text{o}}}{1.33}\right) = 36.9^{\text{o}}$$



For the water to glass interface,

$$\tan\theta_p = \tan\theta_3 = \frac{n_{\text{glass}}}{n_{\text{water}}} = \frac{1.50}{1.33}$$

$$\Rightarrow \theta_3 = 48.4^{\text{o}}$$

The angle between surfaces is $\theta = \theta_3 - \theta_2 = 11.5^{\text{o}}$

Example: Plane- polarized light is incident on a single polarizing disk with the direction of E_0 parallel to the direction of the transmission axis. Through what angle should the disk be rotated so that the intensity in the transmitted beam is reduced by a factor of (a) 3.00, (b) 5.00, (c) 10,0?

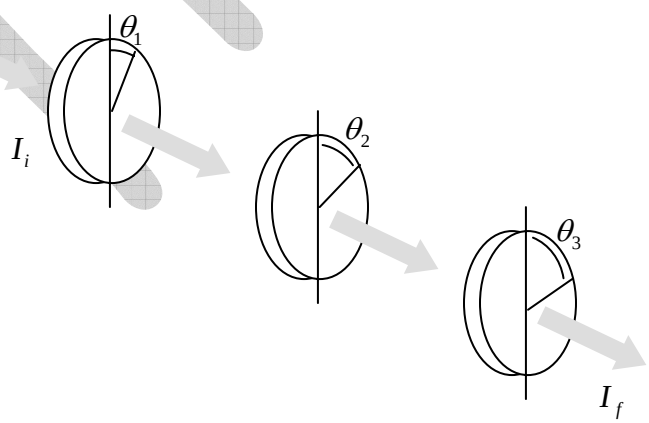
Solution: $I = I_{\max}\cos^2\theta \Rightarrow \quad \theta = \cos^{-1}\sqrt{\dfrac{I}{I_{\max}}}$

(a) $\dfrac{I}{I_{\max}} = \dfrac{1}{3.00} \quad \Rightarrow \quad \theta = \cos^{-1}\sqrt{\dfrac{1}{3.00}} = 54.7^o$

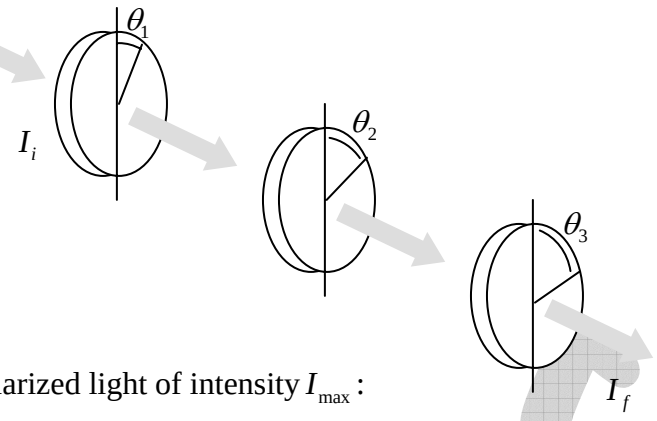
(b) $\dfrac{I}{I_{\max}} = \dfrac{1}{5.00} \quad \Rightarrow \quad \theta = \cos^{-1}\sqrt{\dfrac{1}{5.00}} = 63.4^o$

(c) $\dfrac{I}{I_{\max}} = \dfrac{1}{10.0} \quad \Rightarrow \quad \theta = \cos^{-1}\sqrt{\dfrac{1}{10.0}} = 71.6^o$

Example: In figure below, suppose that the transmission axes of the left and right polarizing disks are perpendicular to each other, also, let center disk be rotated on the common axis with an angular speed ω. Show that if unpolarized light is incident on the left disk with intensity $I_{\max}$ of the beam emerging from right disk is $I = \dfrac{1}{16} I_{\max}\left(1-\cos 4\omega t\right)$



Solution:



For incident unpolarized light of intensity I_{max} :

After transmitting 1st disk: $I = \frac{1}{2} I_{max}$

After transmitting 2nd disk: $I = \frac{1}{2} I_{max} \cos^2\theta$

After transmitting 3rd disk: $I = \frac{1}{2} I_{max} \cos^2\theta \cos^2\left(90^0 - \theta\right)$

Where the angle between the first and second disk is $\theta = \omega t$

Using trigonometric identities $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

And $\cos^2\left(90^0 - \theta\right) = \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

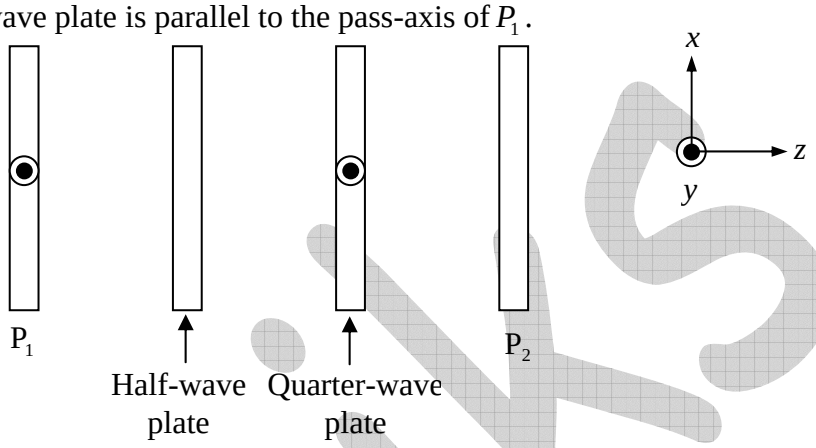
We have $I = \frac{1}{2} I_{max} \left[\frac{(1+\cos 2\theta)}{2} . \frac{(1-\cos 2\theta)}{2}\right]$

$I = \frac{1}{8} I_{max}\left(1-\cos^2 2\theta\right) = \frac{1}{8} I_{max}\left(\frac{1}{2}\right)(1-\cos 4\theta)$

since $\theta = \omega t$, the intensity of the emerging beam is given by

$$I = \frac{1}{16} I_{max}\left[1-\cos\left(4\omega t\right)\right]$$

Example: A half-wave plate and a quarter-wave plate are placed between a polarizer P_1 and an analyzer P_2. All of these are parallel to each other and perpendicular to the direction of propagation of unpolarized incident light (see the figure). The optic-axis of the half-wave plate makes an angle of 30^0 with respect to the pass-axis of P_1 and that of the quarter-wave plate is parallel to the pass-axis of P_1.



(a) Determine the state of polarization for the light after passing through (i) the half-wave plate and (ii) the quarter-wave plate.

(b) What should be the orientation of the pass-axis of P_2 with respect to that of P_1 such that the intensity of the light emerging from P_2 is maximum?

Solution :(a) After passing through HWP the incident ray will split into an o-ray of amplitude $E_0 \sin 30^0 = E_0/2$ and of e-ray of amplitude $E_0 \cos 30^0 = E_0\sqrt{3}/2$. After emerging out of HWP they will have phase difference of π. Superposition of these o- and e-ray will produce linearly polarized light at the output of HWP.

The electric field components of o-ray and e-ray at the output of HWP is

$$\vec{E}_{o-ray} = \left(\frac{E_0}{2}\cos 60^0 \hat{i} - \frac{E_0}{2}\cos 30^0 \hat{k}\right)\sin(\omega t - kz) = \left(\frac{E_0}{4}\hat{i} - \frac{\sqrt{3}E_0}{4}\hat{j}\right)\sin(\omega t - kz)$$

$$\vec{E}_{e-ray} = \left(\frac{\sqrt{3}E_0}{2}\cos 30^0 \hat{i} + \frac{\sqrt{3}E_0}{2}\cos 60^0 \hat{k}\right)\sin(\omega t - kz + \pi)$$

$$= \left(\frac{3E_0}{4}\hat{i} + \frac{\sqrt{3}E_0}{4}\hat{k}\right)\sin(\omega t - kz + \pi)$$

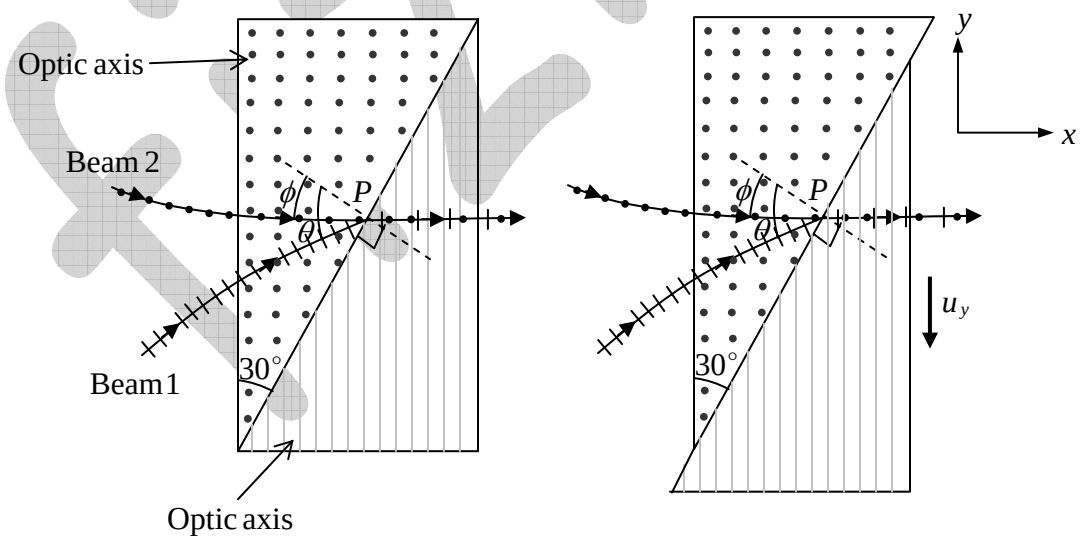
The electric field components of linearly polarized light at the output of HWP is

$$\vec{E} = E_0 \sin(\omega t - kz)\hat{i}$$

The optic axis of QWP is parallel to the P_1 i.e along x-axis. The electric field component of incident linearly polarized light makes zero angle with the optic axis as a result the incident light will simply pass through QWP as a e-ray. Therefore at the output of the QWP light will be linearly polarized with the equation $\vec{E} = E_0 \sin(\omega t - kz)\hat{i}$

(b) The orientation of the pass axis of P_2 should be parallel to the pass axis of P_1 to allow maximum intensity of light to pass through P_2.

Example: Two orthogonally polarized beams (each of wavelength 0.5 μm and with polarization marked in the figure) are incident on a two-prism assembly and emerge along x-direction, as shown. The prisms are of identical material and n_o and n_e are the refractive indices of the o-ray and e-ray, respectively. Use $\sin\phi = \frac{\sin\theta}{3}$, and $n_o = \frac{\sqrt{3}+1}{4}$.



(a) Find the value of θ and n_e. (b) If the right hand side prism starts sliding down with the vertical component of the velocity $u_y = 1\mu m/s$, what would be the minimum time after which the state of polarization of the emergent beam would repeat itself?

Solution:

(a) The beam will propagate as an $o-$ray in the first prism (because the vibrations are perpendicular to the optic axis) and will see the refractive index n_0. When this beam enters the second prism, it will become an $e-$ray and will see the refractive index n_e. For calcite $n_o > n_e$ and the ray will bend away from the normal. Since the optic axis is normal to the plane of, the refracted ray will obey Snell's laws, and the angle of refraction will be given by $n_0 \sin 30^0 = n_e \sin\theta$

We next consider the incidence of a second beam. The beam will propagate as an $e-$ray in the first prism and as an $o-$ray in the second prism. The angle of refraction is now given by $n_e \sin 30^0 = n_0 \sin\phi \quad \Rightarrow \quad n_e \frac{1}{2} = \frac{\sqrt{3}+1}{4} \times \frac{\sin\theta}{3} \quad \Rightarrow \quad n_e = \frac{\sqrt{3}+1}{2} \times \frac{\sin\theta}{3}$

From 1st equation $n_0 \sin 30^0 = n_e \sin\theta \Rightarrow \sin\theta = \frac{n_0}{2n_e} = \frac{\sqrt{3}+1}{4} \times \frac{3\times 2}{\left(\sqrt{3}+1\right)\sin\theta} \times \frac{1}{2}$

$\sin^2\theta = \frac{3}{4} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 60^0$

Hence $n_e = \frac{\sqrt{3}+1}{2} \times \frac{\sin\theta}{3} = \frac{\sqrt{3}+1}{2} \times \frac{\sin 60^0}{3} = \frac{\sqrt{3}+1}{2} \times \frac{\sqrt{3}/2}{3} = \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)$

(b) The state of polarization will repeat if change in phase difference (δ) is π.

The relation between phase difference and path difference (Δ) is $\delta = \frac{2\pi}{\lambda}\Delta$

Where, $\Delta = (n_0 - n_e)d = (n_0 - n_e)\times 10^{-6} \times t$

Thus $\delta = \frac{2\pi}{\lambda}\Delta = \frac{2\pi}{\lambda}(n_0 - n_e)\times 10^{-6} \times t = \pi \quad \Rightarrow \quad t = \frac{1}{4(n_0 - n_e)}$

While $n_0 - n_e = \frac{\sqrt{3}+1}{4} - \frac{\sqrt{3}+1}{4\sqrt{3}} = \frac{1}{2\sqrt{3}}$. Thus $t = \frac{1}{4(n_0 - n_e)} = \frac{1}{4} \times 2\sqrt{3} = \frac{\sqrt{3}}{2} = 0.85\,\text{sec}$

4.8 Analysis of Polarized Light

(i) Linearly polarized (ii) Circularly polarized

(iii) Elliptically polarized (iv) Unpolarized

(v) Mixture of linearly polarized and unpolarized

(vi) Mixture of circularly polarized and unpolarized

(vii) Mixture of elliptically polarized and unpolarized light

If we introduce a Polaroid in the path of the beam and rotate it about the direction of propagation, then one of the following three possibilities can occur:

1. If there is complete extinction at two positions of the polarizer, then the beam is linearly polarized.

2. If there is no variation of intensity, then the beam is unpolarized or circularly polarized or a mixture of unpolarized and circularly polarized light.

We now put a quarter wave plates on the path of the beam followed by the rotating Polaroid. If there is no variation of intensity, then the incident beam is unpolarized. If there is complete extinction at two positions, then the beam is circularly polarized (this is so because a quarter wave plate will transform a circularly polarized light into a linearly polarized light). If there is a variation of intensity (without complete extinction), then the beam is a mixture of unpolarized and circularly polarized light.

3. If there is a variation of intensity (without complete extinction), then the beam is elliptically polarized or a mixture of linearly polarized and unpolarized or a mixture of elliptically polarized and unpolarized light. We now put a quarter wave plate in front of the Polaroid with its optic axis parallel to the pass axis of the Polaroid at the position of maximum intensity. The elliptically Polarized light will transform to a linearly polarized light. Thus, if one obtains two positions of the Polaroid where complete extinction occurs, then the original beam is elliptically polarized. If complete extinction does not occur and the position of maximum intensity occurs at the same orientation as before, the beam is a mixture of unpolarized and linearly polarized light. Finally, if the position of maximum intensity occurs at a different orientation of the Polaroid, the beam is a mixture of elliptically polarized and unpolarized light.

MCQ (Multiple Choice Questions)

Q1. A beam of unpolarized light of intensity I_0 passes through a combination of an ideal polarizer and an idea analyzer with their transmission axes at 60°. What is the intensity of the beam coming out at the other end?

(a) I_0 (b) $\frac{I_0}{2}$ (c) $\frac{I_0}{4}$ (d) $\frac{I_0}{8}$

Q2. A quarter wave plate is placed normally in the path of a plane polarized light beam. The angle between the electric vector and the fast-axis of the plate is 250. The beam coming out of the quarter wave plate will be which one of the following ?

(a) Plane polarized (b) Circularly polarized

(c) Elliptically polarized (d) Unpolarized

Q3. When light passing through rotating Nicol prism is observed, no change in intensity is seen. What inference can be drawn?

(a) The incident light is unpolarized.

(b) The incident light is circularly polarized.

(c) The incident light is uppolarized or circularly polarized.

(d) The incident light is uppolarized or circularly polarized or combination of both.

Q4. When light travels along the optic axis of quartz, which one of the following statements is correct regarding occurrence of optical activity and double refraction?

(a) only optical activity is observed

(b) only double reflection is observed

(c) Both are observed and are at their maximum

(d) None of the two is observed along the optic axis

Q5. A quarter wave plate is placed over a shiny coin. A plane polarizer is placed on top of the quarter wave plate such that the transmission axis of the polarizer is at 45° to the fast axis of the quarter wave plate. How does the shiny coin appear now?

(a) Dark (b) Shiny as before

(c) Shinier than before (d) Coloured

Q6. A circularly polarized beam of light passes through a quarter wave place. The emerging beam is:

(a) plane polarized (b) circularly polarised

(c) elliptically polarized (d) partially polarized

Q7. Which one of the following statements is correct?

Optically active substances are responsible for:

(a) the rotation of the plane of polarization polarized light

(b) producing polarized light

(c) producing brifrigence

(d) converting ordinary light into polarized light

Q8. Match List – I (Phase Difference between Two Similar Superimposed Waves having Mutually Perpendicular States of Polarization and Propagating through the Same Axis) with List – II (Result) and select the correct answer using the code given below the Lists:

List I (Phase Difference between two similar superimposed waves having mutually perpendicular states of polarization and propagating through the same axis)	**List II (Result)**
A. $\theta = 0$	1. Linearly polarized light
B. $\theta = \frac{\pi}{2}$	2. Left circularly polarized light
C. $\theta = \frac{2\pi}{3}$	3. Right circularly polarized light
D. $\theta = \frac{3\pi}{2}$	4. Elliptically polarized light.

Codes:

	A	B	C	D
(a)	1	3	4	2
(b)	4	2	1	3
(c)	1	2	4	3
(d)	4	3	1	2

Q9. Consider two light waves represented by two mutually perpendicular electric field vectors: $E_x = A_x \cos(\omega_x t + \phi_x)$ and $E_y = A_y \sin(\omega_y t + \phi_y)$. Their superposition will result in a plane polarized light, if:

(a) $A_x = A_y, \phi_x = \frac{\pi}{2}, \phi_y = \pi$ (b) $\omega_x = \omega_y, \phi_x = \frac{\pi}{2}, \phi_y = \pi$

(c) $A_x \neq A_y, \omega_x = \omega_y, \phi_x = \phi_y = 0$ (d) $A_x = A_y, \omega_x = \omega_y, \phi_x \neq \phi_y$

Q10. The electric field components of a plane electromagnetic wave are

$E_x = 2E_0 \cos(\omega t - kz); E_y = E_0 \sin(\omega t - kz)$

The state of polarization of the wave will be:

(a) circular (b) plane (c) elliptical (d) unpolarized

Q11. Match List I with List II and select the correct answer using the codes given below

List I	**List II**
A. Double refraction	1. Wollaston
B. Polarizing or Analyzing prism	2. Babinet
C. Double image prism	3. Quartz crystal
D. Elliptically polarized light	4. Polarimeter

Codes:

	A	B	C	D
(a)	3	5	4	2
(b)	2	5	4	3
(c)	3	4	1	2
(d)	2	4	1	3

Q12. When monochromatic light is passed through an analyzing nicol prism, the intensity of the emergent light is found to very between a maximum and a non-zero minimum. Which one of the following cannot represent the polarization characteristics of the incident light?

(a) Mixture of plane-polarised and unpolarised light

(b) Mixture of plane-polarised and circularly-polarised light

(c) Elliptically-polarised light

(d) Plane-polarised light

Q13. A given calcite plate behaves as a half-wave plate for a particular wavelength λ. If the variation of refractive index with λ is negligible, then for a light of wavelength 2λ, the given plate would behave as a:

(a) half-wave-plate (b) quarter-wave plate

(c) plane Polaroid (d) non-polarizing plate

Q14. Two polarizing sheets have their polarizing directions parallel, so that the intensity of the transmitted light is maximum. If the intensity is to drop by one-half, then either of the two sheets must be turned by:

(a) $\pm 30^{o}$ and $\pm 135^{o}$ (b) $\pm 45^{o}$ and $\pm 120^{o}$

(c) $\pm 30^{o}$ and $\pm 120^{o}$ (d) $\pm 45^{o}$ and $\pm 135^{o}$

Q15. If t is the minimum thickness of a quarter wave plate needed to convert plane polarized light of wavelength $480\ nm$ into circular polarized light, then the corresponding thickness of a quarter wave plate for wavelength $600\ nm$ is:

(a) $0.56t$ (b) $0.75\ t$ (c) $1.25\ t$ (d) $1.44\ t$

Q16. The super position of two plane polarized lights in two mutually perpendicular directions given by

$$x = 4\sin\left(\omega t + \frac{\pi}{4}\right) \text{ and } y = 5\sin\left(\omega t + \frac{3}{4}\pi\right)$$

will result in:

(a) plane polarized light (b) unpolarized light

(c) elliptically polarized light (d) circularly polarized light

Q17. If a quarter wave-plate with its fast axis parallel to the surface is inserted into a beam of linearly polarized light oscillating at 45° with fast axis, then the emerging light will be :

(a) linearly polarized (b) vertically polarized

(c) left circularly polarized (d) left elliptically polarized

Q18. The thickness of quarter-wave plate made from a doubly refracting crystal is $6.7\times10^{-5}\ cm$ for a light of wavelength $4800\,\overset{0}{A}$. What is the corresponding thickness of half-wave plate for a light of wavelength $6000\,\overset{0}{A}$?

(a) $16.75\times10^{-5}\ cm$ (b) $13.4\times10^{-5}\ cm$

(c) $10.72\times10^{-5}\ cm$ (d) $8.34\times10^{-5}\ cm$

Q19. A mixture of unpolarized and circularly polarized light is passed through a quarter wave plate and then through an analyzer is rotated:

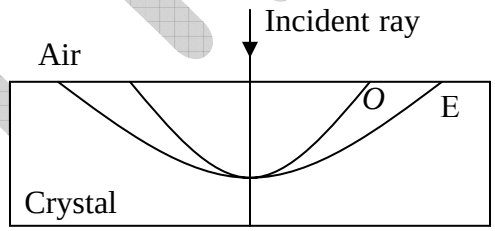
(a) there will be no variation in intensity

(b) there are maxima and minima in intensity with $I_{\min}=0$

(c) there are maxima and minima in intensity with $I_{\min}\neq 0$

(d) the result depends on the orientation of the quarter wave plate

Q20. The given figure illustrates the passage of the ordinary and extra ordinary wavefronts O and E through a uniaxial crystal. The figure represents crystal orientation with its:



(a) optic axis in the plane of incidence and parallel to the crystal surface

(b) optic axis perpendicular to the plane of incidence and parallel to the crystal surface

(c) optic axis in the plane of incidence and perpendicular to the crystal surface

(d) optic axis perpendicular to the plane of incidence and perpendicular to the crystal surface

Q21. An incident light is viewed through a rotating nicol prism. Match List I with List II and select the correct answer by using the codes given below the lists:

List I (Polarization of light incident of rotating nicol prism)	**List II** (Intensity of light as observed through rotating nicol prism)
A. Circularly polarized	1. Intensity changes and vanishes
B. Elliptically polarized	2. Intensity changes but is never completely extinguished
C. Mixture of plane polarized and unpolarized light	3. Intensity does not change
D. Plane polarized light	

Codes:

	A	B	C	D	
(a)	2	3	1	2	3
(b)	1	1	2	3	3
(c)	3	2	2	1	3
(d)	3	2	1	3	1

Q22. A uniaxial birefringent crystal is cut to form a parallel plate with its optic axis parallel to the front face. For such a retarder plate of refractive indices n_0 and n_e, with thickness d, the expression for the relative phase difference, $\Delta\phi$, for the wavelength λ is given by:

(a) $\dfrac{|n_0 - n_e|d}{\lambda}$ (b) $\dfrac{\pi|n_0 - n_e|d}{\lambda}$

(c) $\dfrac{\pi|n_0 - n_e|d}{2\lambda}$ (d) $\dfrac{2\pi|n_0 - n_e|d}{\lambda}$

Q23. Group I contains x- and y- components of the electric field and Group II contains the type of polarization of light.

	Group I	**Group II**
P.	$E_x = \frac{E_0}{\sqrt{2}}\cos(\omega t + kz)$, $E_y = E_0 \sin(\omega t + kz + \pi)$	1. Circularly Polarized
Q.	$E_x = E_0 \sin\left(\omega t + kz + \frac{\pi}{6}\right)$, $E_y = E_0 \sin\left(\omega t + kz - \frac{\pi}{3}\right)$	2. Elliptically Polarized
R.	$E_x = E_0 \sin\left(\omega t + kz + \frac{3\pi}{4}\right)$, $E_y = E_0 \sin\left(\omega t + kz - \frac{\pi}{4}\right)$	3. Linearly Polarized

The correct set of matches is

(a) $P \to 1; Q \to 2; R \to 3$ (b) $P \to 1; Q \to 3; R \to 2$

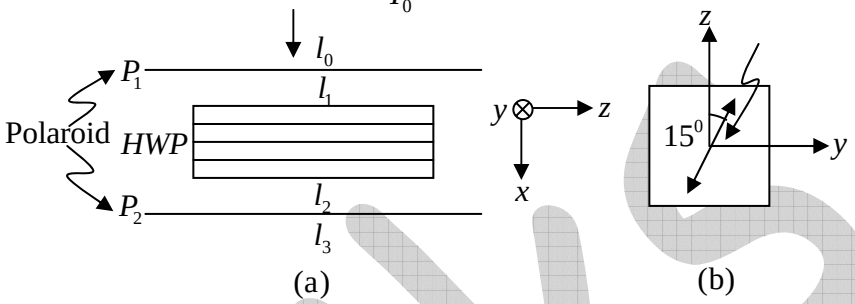
(c) $P \to 2; Q \to 1; R \to 3$ (d) $P \to 2; Q \to 3; R \to 1$

Q24. The thickness of a quarter wave plate made of quartz for wavelength $\lambda = 6000\,\overset{0}{A}$, refractive indices $\mu_e = 1.553$ and $\mu_0 = 1.543$ is:

(a) $1.25\times10^{-3}\ cm$ (b) $0.15\times10^{-3}\ cm$

(c) $1.50\times10^{-3}\ cm$ (d) $2.50\times10^{-3}\ cm$

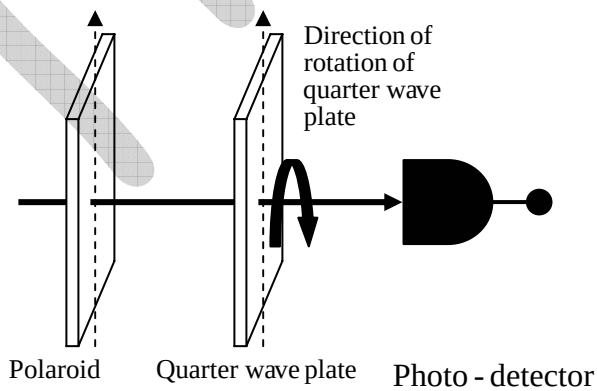
Q25. A HWP is introduced between two crossed Polaroids P_1 and P_2. The optic axis makes an angle of 15° with the pass axis of P_1 as shown in Fig. (a) and (b). If an unpolarized beam of intensity I_0 is normally incident on P_1 and if I_1, I_2, and I_3 are the intensities after P_1, after HWP, and after P_2, respectively, $\frac{I_3}{I_0}$ is



(a) (b)

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) $\frac{1}{8}$

Q26. A half-wave plate is placed in between a polarizer and a photo-director. When the optic axis of the quarter-wave plate is kept initially parallel to the pass axis of the polarizer and perpendicular to the direction of light propagation. The intensity of light passing through the quarter-wave plate is measured to be I_0 (see figure). If the quarter wave plate is now rotated by 45^0 about an axis parallel to the light propagation, what would be the intensity of the emergent light measured by the photo-director?



(a) $\frac{I_o}{\sqrt{2}}$ (b) $\frac{I_0}{2}$ (c) $\frac{I_0}{2\sqrt{2}}$ (d) I_0

Q27. Unpolarized light of intensity I pass through two Polaroid sheets. The axis of the first is vertical, and that of the second is at 30^0, to the vertical, what fraction of the incident light is transmitted?

(a) $\frac{3}{4}I$ (b) $\frac{1}{2}I$ (c) $\frac{1}{4}I$ (d) $\frac{3}{8}I$

MSQ (Multiple Select Questions)

Q29. A beam of light passes through a quarter wave plate and the emergent beam is circularly polarized. For this to happen,

(a) electric vector should be parallel to the optic axis

(b) incident beam should be plane polarized

(c) incident beam should be unpolarized

(d) electric vector should make an angle of 45° with the optic axis

Q30. Consider the following statements:

A beam of light passes through a quarter wave plate and the emergent beam is circularly polarized. For this to happen,

(a). incident beam should be plane polarized

(b). electric vector should make an angle of 45° with the fast axis

(c). incident beam should be unpolarized

(d). electric vector should be parallel to the fast axis

Q31. Consider the following statements:

When plane polarized monochromatic light passes through an optically active substance,

(a). its plane of polarization rotates clockwise for all substances.

(b). its plane of polarization rotates anticlockwise for all substances.

(c). the angle of rotation of plane of polarization is proportional to the distance traveled inside the substance.

(d) the angle of rotation of plane of polarization is inversely proportional to the distance traveled inside the substance.

Q32. When an unpolarised light beam passes through a double refracting medium, it splits up into two beams called ordinary ray and extraordinary ray.

Consider the following statements.

(a). Intensities of the both rays are equal.

(b). Refractive index of the ordinary ray remains constant.

(c). Refractive index of the extraordinary ray does not remain constant.

(d). Both the rays are polarized.

Q33. Group I contains x- and y- components of the electric field and Group II contains the type of polarization of light.

(1) $E_x = \frac{E_0}{\sqrt{2}}\cos(\omega t + kz)$

$E_y = E_0 \sin(\omega t + kz)$

(2) $E_x = E_0 \sin(\omega t + kz)$

$E_y = E_0 \cos(\omega t + kz)$

(3) $E_x = E_1 \sin(\omega t + kz)$

$E_y = E_2 \sin(\omega t + kz)$

(4) $E_x = E_0 \sin(\omega t + kz)$

$E_y = E_0 \sin\left(\omega t + kz + \frac{\pi}{4}\right)$

Which of the following statements are correct

(a) Resultant of (1) leads to elliptically polarized

(b) Resultant of (2) leads to circularly polarized

(c) Resultant of (3) leads to linarly polarized

(d) Resultant of (4) leads to Left circularly polarized

NAT (Numerical Answer Type)

Q34. Unpolarised light of intensity I_0 gets polarized emerges from a polarizer. It then passes through an analyzer. The intensity of light emerging from the analyzer is $\frac{I_0}{8}$. The angle between the polarizing direction of the polarizer and analyzer is ……. degree

Q35. A quarter wave plate is designed for a wavelength of $600\,nm$. The difference in refractive indices for the electric components along the fast and the slow axes is 0.2. The geometrical thickness of the plate will be ……. mm

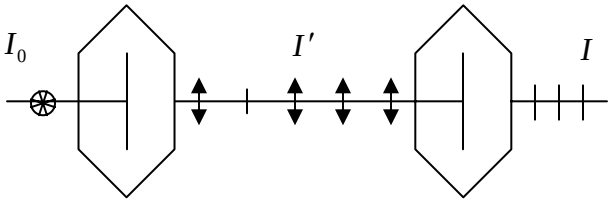
Q36. Unpolarized light in incident on a glass plate having refractive index 1.5. The angle of incidence at which the plane polarized light is obtained, is …….. degree

Q37. The thickness of a quarter wave plate made of quartz for wavelength $\lambda = 5000\,\overset{0}{A}$, refractive indices $\mu_e = 1.553$ and $\mu_0 = 1.543$ is_____ $\times 10^{-3}\,cm$.

Solution-MCQ (Multiple Choice Questions)

Ans. 1: (d)

Solution:



When an unploarized of intensity I_0 passes through a polarizer then intensity becomes half i.e., $I' = \frac{1}{2} I_0$ -------(i)

Now by Maulus law $I = I' \cos^2 \theta$ where $\theta = 60^0$

$$\Rightarrow \quad I = \frac{I_0}{2} \cos^2 60^0 = \frac{I_0}{2} \times \left(\frac{1}{2}\right)^2 = \frac{I_0}{8}$$

Ans. 2: (c)

Solution: The quarter wave plate is a transparent plate which produces a path difference of $\frac{\lambda}{4}$ for ordinary and extraordinary ray.

If the polarized light makes an angle of 25° with optic axis then amplitude of extraordinary ray is $E_0 \cos\phi$ and ordinary ray is $E_0 \sin\phi$

Since, their amplitudes are same hence, they from elliptical polarized light after emerging from the plate.

Ans. 3: (d)

Ans. 4: (a)

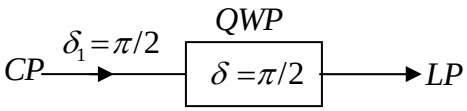
Solution: The rotation of plane of vibration when a polarized light passes through an optical active medium is called optical activity.

Ans. 5: (d)

Solution: A quarter wave plate kept at an angle of 45^0 with optic axis produces plane polarized light hence, coin appears colored.

Ans. 6: (a)

Solution:



When a circularly polarized beam passes through a quarter wave plate, it becomes plane polarized.

Because the quarter wave plate is plate which can produces the path difference of $\frac{\lambda}{4}$ and phase difference of $\frac{\pi}{2}$ between ordinary and extra ordinary ray.

The total phase difference after QWP is $\delta = \delta_1 + \delta_2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

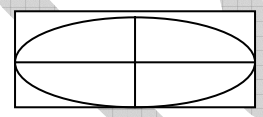
Thus output light will be linearly polarized

Ans. 7: (a)

Solution: When a plane polarized light enters in an optically active substance the plane polarized light is rotated clockwise or anticlockwise depending on nature of the substance.

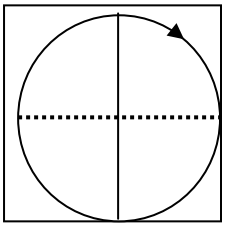
Ans. 8: (a)

Solution:



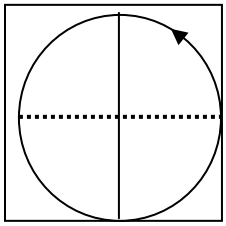
elliptical

$\phi = \frac{2\pi}{3}$



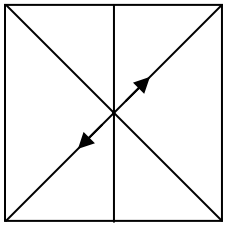
left circular

$\phi = \frac{2\pi}{3}$



right circular

$\theta = \frac{\pi}{2}$



plane polarised

$\theta = 0$

Ans. 9: (b)

Solution: Te ordinary light is a wave in which its both components E and B vibrate perpendicularly to the direction of the light. If E vibrate in a plane the light is said to plane polarized light.

The two perpendicular vibrating light waves from a plane polarized light if their frequencies are equal $\omega_x = \omega_y$ and have a phase difference of zero. .

$$E_x = A_x \cos(\omega_x t + \phi_x) = A_x \sin\left(\omega_x t + \phi_x + \frac{\pi}{2}\right) \text{ and } \quad E_y = A_y \sin(\omega_y t + \phi_y)$$

The phase difference is $\delta = \phi_x + \frac{\pi}{2} - \phi_y = 0 \quad \Rightarrow \phi_x - \phi_y = -\frac{\pi}{2}$

This is possible when $\phi_x = \frac{\pi}{2} + \phi_y = \pi$.

Ans. 10: (c)

Solution: We have $E_x = 2E_0 \cos(\omega t - kz)$

$$\Rightarrow \quad \frac{E_x}{2E_0} = \cos(\omega t - kz) \quad \text{-------(i)}$$

and $\quad E_y = E_0 \sin(\omega t - kz)$

$$\Rightarrow \quad \frac{E_y}{E_0} = \sin(\omega t - kz) \quad \text{--------(ii)}$$

squaring equations. (i) and (ii) and adding, we get

$$\frac{E_x^2}{4E_0^2} + \frac{E_y^2}{E_0^2} = 1$$

This represents equation of ellipse.

Hence, it is elliptically polarized.

Ans. 11: (c)

Ans. 12: (d)

Solution: When a plane polarized light passes through an analyzer and analyzer, is rotating then intensity becomes maximum and a zero minimum twice for one complete rotation of the analyzer.

Ans. 13: (b)

Solution: A plate which produces a path difference of $\left(\frac{\lambda}{4}\right)$ between ordinary and extraordinary rays is called quarter-wave plate. If this path difference is $\left(\frac{\lambda}{2}\right)$, then it is called Half-wave plate.

For quarter-wave plate $t_{1/4} = \frac{\lambda}{4(\mu_e - \mu_0)}$

For half-wave plate $t_{1/2} = \frac{\lambda}{2(\mu_e - \mu_0)}$

If t is width of plate then $(\mu_0 - \mu_e)t = \left(\frac{\lambda}{2}\right)$

Again, $(\mu_0 - \mu_e)t = \frac{2\lambda}{4}$

Hence, for wavelength 2λ the plate will behave as a quarter-wave plate.

Ans. 14: (d)

Solution: According to Malaus law intensity is proportional the square of the cosine of angle between and polarizer, i.e. $I \propto \cos^2\theta$ Here $\cos^2\theta = \frac{1}{2}$

$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}} \Rightarrow \theta = \pm 45^o, \pm 135^o$.

Ans. 15: (c)

Solution: If λ is wavelength of a monochromatic light thickness of the quarter wavelength is as

Let $t_1 = \frac{\lambda_1}{4(\mu_0 - \mu_c)}$ and $t_2 = \frac{\lambda_2}{4(\mu_0 - \mu_c)}$

Thus, $t_2 = t_1\frac{\lambda_2}{\lambda_1} = t_1\frac{600}{480} = t\left(\frac{5}{4}\right) = 1.25\ t$

Ans. 16: (c)

Solution: Let $y = a\sin(\omega t + \phi)$ and $x = b\sin\omega t$

Given $x = 4\sin\left(\omega t + \frac{\pi}{4}\right)$ and $y = 5\sin\left(\omega t + \frac{3\pi}{4}\right)$

where, $a = 5, b = 4,$ and $\phi = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$

Since $a \neq b$ and $\phi = \frac{\pi}{2}$, so resultant will be elliptically polarized.

Ans. 17: (c)

Solution: As the surface of QWP, the electric field components are

$$E_x = (E_0\cos\theta)\sin(\omega t)\hat{x} = \frac{E_0}{\sqrt{2}}\sin\omega t\,\hat{x}$$

$$E_y = (E_0\cos\theta)\sin(\omega t)\hat{y} = \frac{E_0}{\sqrt{2}}\sin\omega t\,\hat{y}$$

Thus at the output of QWP, the E - field components are

$$E_x = \frac{E_0}{\sqrt{2}}\sin(\omega t) \text{ and } E_y = \frac{E_0}{\sqrt{2}}\sin\left(\omega t - \frac{\pi}{2}\right)$$

Thus the output will be left circularly polarized light.

The quarter wave plate for which the path difference between ordinary ray and extra ordinary ray is $\frac{\lambda}{4}$ or phase difference is $\left(\frac{\pi}{2}\right)$.

Ans. 18: (a)

Solution: If n_e and n_0 are refractive indices of extraordinary and ordinary rays in case of doubly refraction then thickness of wave plate are given as

$(n_e - n_0) =$ (in term of wavelength)

For half wavelength $t_1 = \frac{\lambda}{2}$ --------(i)

For quarter wavelength $t_2 = \frac{\lambda}{4}$ --------(ii)

By equation (i) and (ii) half wave plate $= 2\times$ thickness of quarter wave plate

So, $$(n_e - n_0)\frac{t_1}{4} = \frac{\lambda'}{4} \qquad \text{--------(iii)}$$

and $$(n_e - n_0)t_{1/2} = \frac{\lambda''}{2} \qquad \text{--------(iv)}$$

Dividing equation (iv) by (iii), we get

$$t_{1/2} = \left(\frac{\lambda''}{2}\right)\left(\frac{4}{\lambda'}\right)\cdot t_{1/4}$$

$$\Rightarrow \quad t_{1/2} = \left(\frac{6000}{2}\right)\left(\frac{4}{4800}\right)\times 6.7\times 10^{-5} = 16.75\times 10^{-5}\ cm$$

Ans. 19: (c)

Solution: When unpolarized with circular polarized light is passed through a quarter wave plate, it becomes a plane polarized light and hence if it again passes through an analyzer the intensity becomes maximum and minimum due to the rotation of analyzer but minima never become zero.

Ans. 20: (d)

Solution: Extra ordinary ray travels with different velocities whereas the ordinary ray travels with same velocity in all directions. The velocities of extra ordinary and ordinary rays are same along the optic axis. From the figure, it is clear that along the incident ray, their velocities are equal. Hence, the optic axis is along the incident ray.

Ans. 21: (c)

Solution:

(i) Circularly polarized light	Intensity does not change
(ii) Unpolarised light	Intensity does not change
(iii) Elliptically polarised	Intensity changes but never completely extinguished
(iv) Mixture of plane polarized	Intensity changes but never completely extinguished
(v) Plane polarized light	Intensity changes and vanishes

Ans. 22: (d)

Solution: The path difference $\delta = (n_0 - n_e)d$

$\Rightarrow$ The phase difference $\phi = \frac{2\pi}{\lambda}\delta = \frac{2\pi|n_0 - n_e|d}{\lambda}$

Ans. 23: (c)

P. $E_x = \frac{E_0}{\sqrt{2}}\cos(\omega t + kz)$ *and* $E_y = E_0 \sin(\omega t + kz + \pi)$

The phase difference between E_x and E_y is $\frac{\pi}{2}$ with different amplitude. Therefore the resultant will be elliptically polarized.

Q. $E_x = E_0 \sin\left(\omega t + kz + \frac{\pi}{6}\right)$ *and* $E_y = E_0 \sin\left(\omega t + kz - \frac{\pi}{3}\right)$

The phase difference between E_x and E_y is $\frac{\pi}{2}$ with same amplitude. Therefore the resultant will be circularly polarized.

R. $E_x = E_1 \sin\left(\omega t + kz + \frac{3\pi}{4}\right)$ *and* $E_y = E_2 \sin\left(\omega t + kz - \frac{\pi}{4}\right)$

The phase difference between E_x and E_y is π with different amplitude. Therefore the resultant will be linarly polarized.

Ans. 24: (c)

Solution: The plate which produce a path difference of $\left(\frac{\lambda}{4}\right)$ for extraordinary and ordinary rays is called quarter wave plate.

Hence , $(\mu_e - \mu_0)t = \frac{\lambda}{4} \Rightarrow$ $(1.533 - 1.543)t = \frac{6000\times10^{-10}}{4}m$

$\Rightarrow$ $0.010t = \frac{6000\times10^{-10}}{4}m$ $\Rightarrow$ $t = 1.5\times10^{-3}cm$

Ans. 25: (d)

Solution: Unpolarized light of Intensity I_0 incident on Polaroid P_1, the intensity I_1 will be $\frac{I_0}{2}$.

HWP introduce a phase difference of π, output of HWP will be linearly polarized with no change in the intensity at the output of HWP. The electric field vector of linearly polarized light will make angle 30° with the pass axis of the second polarizer.

According to Malu's law intensity at output will be

$$I_3 = \frac{I_0}{2}\cos^2\theta = \frac{I_0}{2}\cos^2(\pi/6) = \frac{I_0}{8}$$

Ans. 26: (d)

Solution: After passing through half wave plate the plane polarized light of intensity I_0 will remain in early polarized rotated at 45^0 but the intensity is driven by Malu's law

$$I = I_0\cos^2(\theta) = I_0\cos^2(45) = \frac{I_0}{2}$$

Ans. 27: (d)

Solution: The average value of the cosine squard function is one-half, so the first polarizer transmits $\frac{1}{2}I$ of the light.

The second transmits $\cos^2\left(30^0\right) = \frac{3}{4}$

$$\therefore \quad I_f = \frac{1}{2}\times\frac{3}{4}I = \frac{3}{8}I$$

MSQ (Multiple Select Questions)

Ans. 29: (b) and (d)

Ans. 30: (a) and (b)

Solution: A transparent plate which can produce a path difference of one quarter of wavelength i.e., $\frac{\lambda}{4}$ between extraordinary and ordinary ray is called quarter wave plate.

For this to happen

(i) incident beam should be plane polarized

(ii) electric vector E should make an angle of 45° with the fast axis

Ans. 31: (d)

Solution: When a plane polarized light passes through an special type of substance along the optic axis the plane of polarization is rotated clockwise or anticlockwise depending on the nature of the substance. This special type of substance is called optically active substance and this phenomenon is known as optic activity.

Ans. 32: (b) and (c)

Solution: when an unpolarised light beam passes through a double refracting medium it splits into two beams viz extraordinary ray which has different velocity in different directions and other is ordinary ray whose velocity in all directions is same.

Both these are plane polarized and the plate of polarizations of these are perpendicular to each other.

Ans. 33: (a), (b) and (c)

Solutions:

1. $E_x = \frac{E_0}{\sqrt{2}}\cos(\omega t + kz)$ *and* $E_y = E_0 \sin(\omega t + kz)$

The phase difference between E_x and E_y is $\frac{\pi}{2}$ with different amplitude. Therefore the resultant will be elliptically polarized.

2. $E_x = E_0 \sin(\omega t + kz)$ *and* $E_y = E_0 \cos(\omega t + kz)$

The phase difference between E_x and E_y is $\frac{\pi}{2}$ with same amplitude. Therefore the resultant will be circularly polarized.

3. $E_x = E_1 \sin(\omega t + kz)$ *and* $E_y = E_2 \sin(\omega t + kz)$

The phase difference between E_x and E_y is 0 with different amplitude. Therefore the resultant will be linarly polarized.

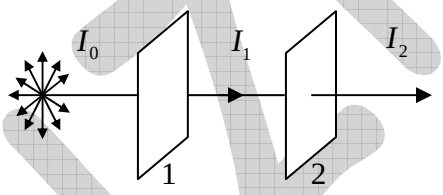
4. $E_x = E_0 \sin(\omega t + kz)$ *and* $E_y = E_0 \sin\left(\omega t + kz + \frac{\pi}{4}\right)$

The phase difference between E_x and E_y is $\frac{\pi}{4}$ with same amplitude. Therefore the resultant will be elliptically polarized.

NAT (Numerical Answer Type)

Ans. 34: 60°

Solution:



$2 \rightarrow$ Analyzer

$1 \rightarrow$ Polarizer

Since, average of $\cos^2\theta$ is $\frac{1}{2}$ so if I_0 is intensity of unpolarised light then intensity of polarized coming out from the polarizer is $\frac{1}{2}$

Thus, $I_1 = \frac{1}{2} I_0$ ------(i)

By Malaus law, the intensity of light coming out from the analyzer

$I_2 = I_p \cos^2\theta$ ------(ii)

By equation (i) & (ii), we get

$$I_2 = \frac{1}{2} I_0 \cos^2\theta \qquad \text{------(iii)}$$

By question $I_2 = \frac{1}{8} I_0$ -------(iv)

By equation (iii) and (iv) $\frac{1}{8} I_0 = \frac{1}{2} I_0 \cos^2\theta$

$\Rightarrow \quad \cos^2\theta = \frac{1}{4} \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \theta = 60^0$

Ans. 35: 750

Solution: A quarter wave plate is plate for which the difference of path between extra ordinary and ordinary ray is $\frac{\lambda}{4}$.

$$(\mu_e - \mu_o)t = \frac{\lambda}{4}$$

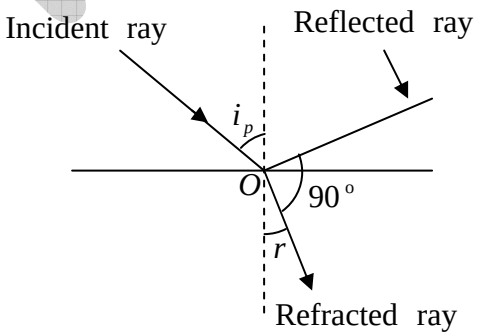
Hence, $\mu_e - \mu_o = 0.2, \lambda = 600\times10^{-9}$

So, $t = \frac{\lambda}{4(\mu_e - \mu_o)}$

$\Rightarrow \quad t = \frac{600\times10^{-9}}{4(0.2)} m = 750\ nm$

Ans. 36: 57

Solution:



By Brewster's law:

$\tan i_p = n$

$\Rightarrow \quad \tan i_p = 1.5$ (given)

$\Rightarrow \quad i_p = 57^0$

Ans. 37: 1.25

Solution: The plate which produce a path difference of $\left(\frac{\lambda}{4}\right)$ for extraordinary and ordinary rays is called quarter wave plate.

Hence , $(\mu_e - \mu_0)t = \frac{\lambda}{4}$

$\Rightarrow \quad (1.533 - 1.543)t = \frac{5000 \times 10^{-10}}{4} m$

$\Rightarrow \quad 0.010t = \frac{5000 \times 10^{-10}}{4} m$

$\Rightarrow \quad t = 1.25 \times 10^{-3} cm$

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